

Spacelike spherically symmetric CMC hypersurfaces in Schwarzschild spacetimes (I): Construction

Kuo-Wei Lee and Yng-Ing Lee

Abstract

We solve spacelike spherically symmetric constant mean curvature (SS-CMC) hypersurfaces in Schwarzschild spacetimes and analyze their asymptotic behavior near the coordinate singularity $r = 2M$. Furthermore, we join SS-CMC hypersurfaces in the Kruskal extension to obtain complete ones and discuss the smooth properties.

1 Introduction

The Schwarzschild spacetime is the simplest model of a universe containing a star. Its metric is a solution of the vacuum Einstein equations, and is spherically symmetric, asymptotically flat, and Ricci flat. A more remarkable fact is that the Schwarzschild metric is the only spherically symmetric vacuum solution of the Einstein equations.

Spacelike constant mean curvature (S-CMC) hypersurfaces in spacetimes have been considered important and interesting objects in studying the dynamics of spacetime and in general relativity. We refer to [8] for more discussions on the importance of S-CMC hypersurfaces. From the viewpoint of geometry, a S-CMC hypersurface in spacetimes has extremal surface area with fixed enclosed volume [1]. This property is similar to that of a compact CMC hypersurface in Euclidean spaces.

In this paper, we study spacelike spherically symmetric constant mean curvature (SS-CMC) hypersurfaces in Schwarzschild spacetimes and Kruskal extension. We solve SS-CMC hypersurfaces in both exterior and interior of the Schwarzschild spacetime, and then analyze their asymptotic behavior, especially at $r = 2M$. The Kruskal extension is an analytic extension of the Schwarzschild spacetime. When SS-CMC hypersurfaces are mapped to the Kruskal extension, we find relations between SS-CMC hypersurfaces in exterior and interior

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such that they can be joined smoothly. These statements can be seen in Theorem 1–5. Furthermore, we get all complete SS-CMC hypersurfaces in the Kruskal extension in Theorem 6.

Our motivation on studying SS-CMC hypersurfaces is on one hand that they are easier to deal with and have explicit expressions, and on the other hand that these examples can serve as barrier functions for the general non-symmetric cases. We hope that a deep understanding of these solutions can help us to find right formulation of other general questions in the Schwarzschild spacetime such as Dirichlet problem and etc. After we finished the results in this paper, we found that the problem was also studied by Brill, Cavallo, and Isenberg in [1], and Malec, and Ó Murchadha in [6, 7]. However, the approaches are quite different. Our viewpoint is purely geometrical and the explicit formula derived in this paper has the advantage on verifying foliation properties conjectured in [6]. This part will appear in a forthcoming paper [5].

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The organization of this paper is as follows. We first give a brief summary of the Schwarzschild spacetime and the Kruskal extension in section 2. A good reference for this part is Wald’s book [9]. In sections 3–5, we study SS-CMC hypersurfaces in each region and analyze their asymptotic behavior, especially at $r = 2M$. How to glue these solutions into complete and smooth SS-CMC hypersurfaces are discussed in section 6.

2 The Kruskal extension

The Schwarzschild spacetime, denoted by S , has a metric

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (1)$$

We often write $h(r) = 1 - \frac{2M}{r}$. The metric (1) is not defined at $r = 0$ and $r = 2M$, and looks singular at both places. But in fact, the Schwarzschild spacetime is nonsingular at $r = 2M$. It is only a coordinate singularity, which is caused merely by a breakdown of the coordinates. There is a larger spacetime including the Schwarzschild spacetime as a proper subset and it has a smooth metric, especially for points corresponding to $r = 2M$. Such an analytic extension was obtained by Kruskal in 1960.

Proposition 1. [4, 9] *The Schwarzschild metric can be written as*

$$\begin{aligned} ds^2 &= \frac{16M^2 e^{-\frac{r}{2M}}}{r} (-dT^2 + dX^2) + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \\ &= \frac{16M^2 e^{-\frac{r}{2M}}}{r} dU dV + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \end{aligned} \quad (2)$$

where

$$\begin{cases} (r - 2M)e^{\frac{r}{2M}} = X^2 - T^2 = VU \\ \frac{t}{2M} = \ln \left| \frac{X+T}{X-T} \right| = \ln \left| \frac{V}{U} \right|. \end{cases} \quad (3)$$

The metric (2) is nonsingular at $r = 2M$.

A spacetime diagram for the Kruskal extension is shown in Figure 1. Each point in the Kruskal plane represents a sphere. There is one-to-one and onto correspondence from the region I to the Schwarzschild exterior $r > 2M$, and from the region II to the Schwarzschild interior $0 < r < 2M$. The whole Kruskal extension is the union of regions I, II, I', and II', where regions I' and II' are exterior and interior of another Schwarzschild spacetime, respectively.

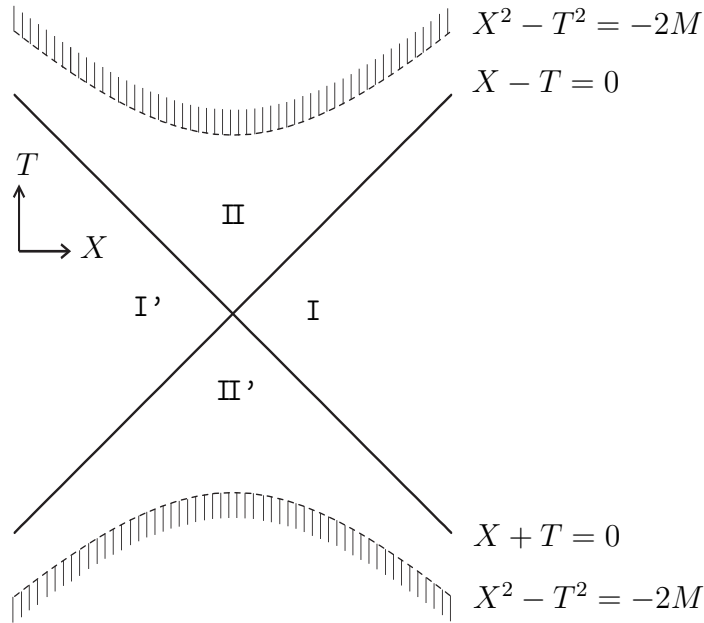


Figure 1: The Kruskal extension of a Schwarzschild spacetime.

From (3), we know that each $r = \text{constant}$ in the Schwarzschild spacetime is a hyperbola in the Kruskal extension, and each $t = \text{constant}$ in the Schwarzschild spacetime is two half-lines starting from the origin in the Kruskal extension. Images of $r = \text{constant}$ and $t = \text{constant}$ under the correspondence are illustrated in Figure 2.

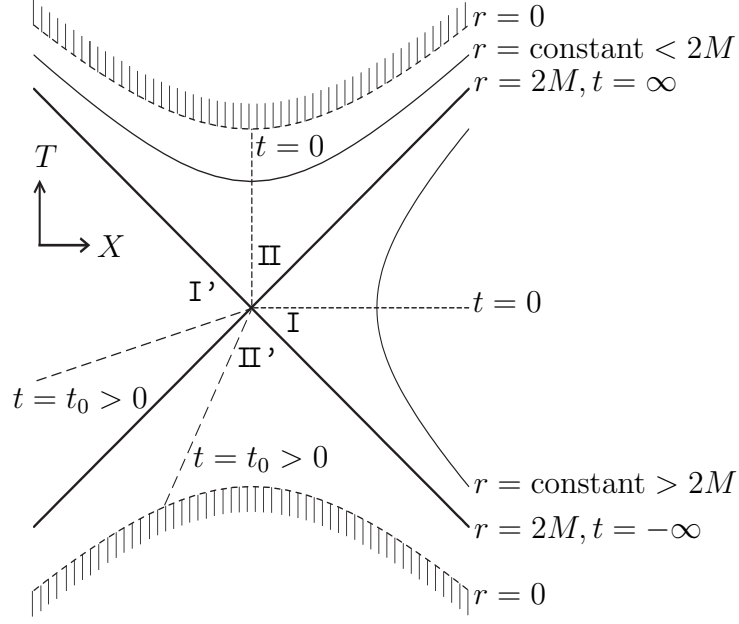


Figure 2: Level sets $r = \text{constant}$ and $t = \text{constant}$.

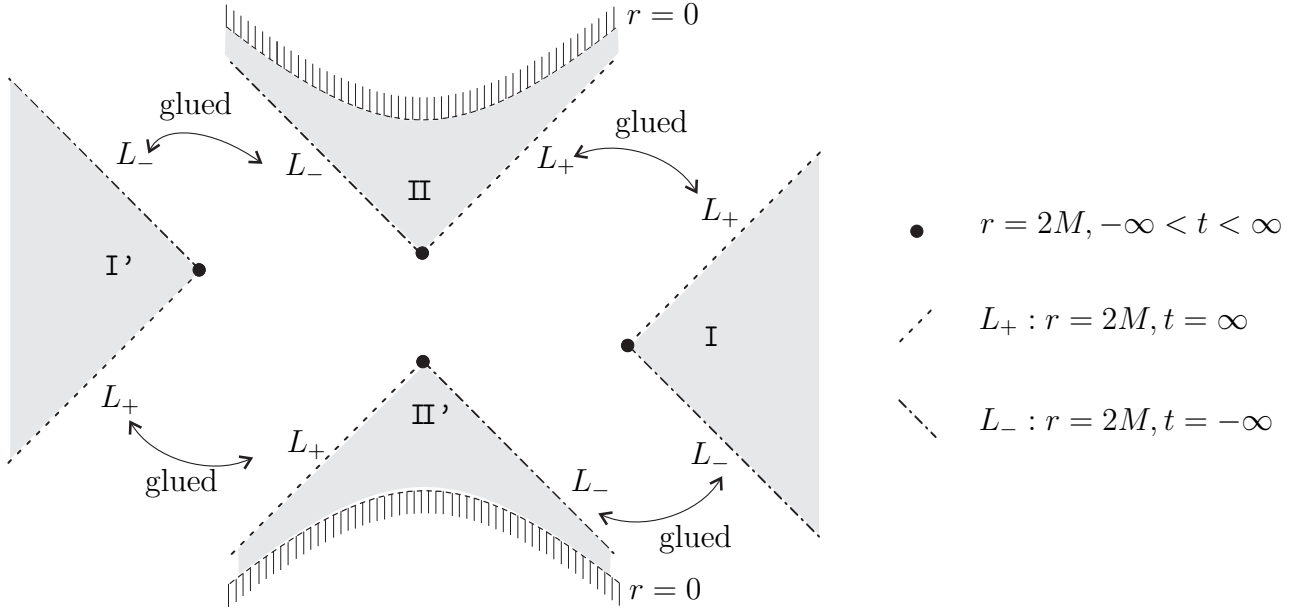


Figure 3: The gluing of Schwarzschild exteriors and interiors.

Now we explain how the Schwarzschild exterior and interior change as they map into the Kruskal extension. The boundary $r = 2M$, $-\infty < t < \infty$ of the Schwarzschild exterior and interior blow down to the origin in the Kruskal extension. On the other hand, $r = 2M$, $t = \infty$ and $r = 2M$, $t = -\infty$ blow up to half-lines L_+ and L_- in the Kruskal extension, respectively. The L_+ of I is glued to the L_+ of II, and the L_- of II is glued to the L_- of I', and so on. Moreover, $r = 0$ is mapped to the hyperbola $X^2 - T^2 = -2M$ in the Kruskal

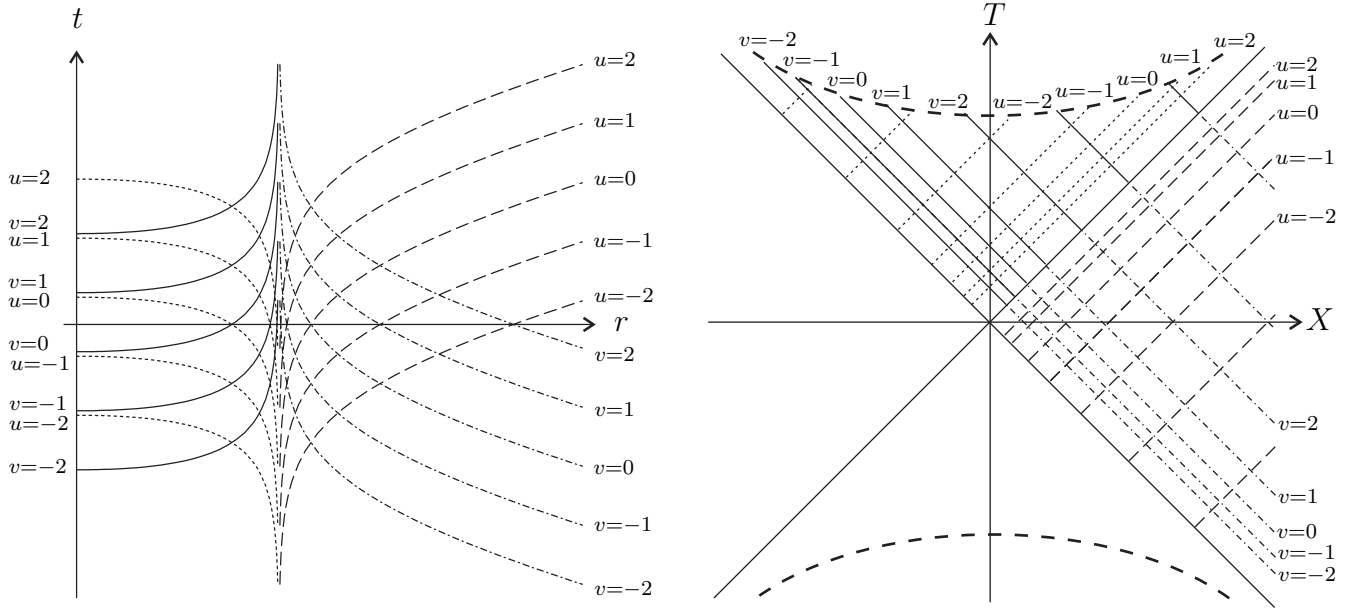


Figure 4: Null geodesics in the Schwarzschild spacetime and Kruskal extension.

extension. This identification is pictured in Figure 3.

The idea to the construction of the Kruskal extension is using null geodesics. When omitting the spherically symmetric part and solving null geodesics in t - r plane, we can define null coordinates u, v by

$$u = t - (r + 2M \ln |r - 2M|) \quad \text{and} \quad v = t + (r + 2M \ln |r - 2M|).$$

These coordinate curves are mapped to $\pm 45^\circ$ straight lines in the Kruskal extension. Figure 4 presents $u = \text{constant}$ and $v = \text{constant}$ in the Schwarzschild spacetime and Kruskal extension. Furthermore, we can define null coordinates (U, V) in the Kruskal extension by

	Region I	Region II	Region I'	Region II'
U	$e^{-\frac{u}{4M}}$	$-e^{-\frac{u}{4M}}$	$-e^{-\frac{u}{4M}}$	$e^{-\frac{u}{4M}}$
V	$e^{\frac{v}{4M}}$	$e^{\frac{v}{4M}}$	$-e^{\frac{v}{4M}}$	$-e^{\frac{v}{4M}}$

Direct computation from (3) gives the relations between (X, T) and (r, t) as follows:

$$\begin{aligned}
\text{In region I,} \quad X &= \frac{\sqrt{r-2M}(e^{\frac{r+t}{4M}} + e^{\frac{r-t}{4M}})}{2} & \text{and} & \quad T = \frac{\sqrt{r-2M}(e^{\frac{r+t}{4M}} - e^{\frac{r-t}{4M}})}{2}. \\
\text{In region II,} \quad X &= \frac{\sqrt{2M-r}(e^{\frac{r+t}{4M}} - e^{\frac{r-t}{4M}})}{2} & \text{and} & \quad T = \frac{\sqrt{2M-r}(e^{\frac{r+t}{4M}} + e^{\frac{r-t}{4M}})}{2}. \\
\text{In region I}', \quad X &= -\frac{\sqrt{r-2M}(e^{\frac{r+t}{4M}} + e^{\frac{r-t}{4M}})}{2} & \text{and} & \quad T = -\frac{\sqrt{r-2M}(e^{\frac{r+t}{4M}} - e^{\frac{r-t}{4M}})}{2}. \\
\text{In region II}', \quad X &= -\frac{\sqrt{2M-r}(e^{\frac{r+t}{4M}} - e^{\frac{r-t}{4M}})}{2} & \text{and} & \quad T = -\frac{\sqrt{2M-r}(e^{\frac{r+t}{4M}} + e^{\frac{r-t}{4M}})}{2}.
\end{aligned}$$

In this article, we always take ∂_T as future directed timelike vector field. In region I, the vector ∂_T points to the direction of increasing t , while in region II it points to the direction of decreasing r . On the other hand, ∂_T points to the direction of decreasing t in region I' and points to the direction of increasing r in region II'.

3 SS-CMC solutions in region I

A vector v is *spacelike* if $\langle v, v \rangle > 0$, *null* if $\langle v, v \rangle = 0$, and *timelike* if $\langle v, v \rangle < 0$. Given a smooth function F on the Schwarzschild spacetime (S, ds^2) with ds^2 as in (1), denote a level set of F by $\Sigma = \{x \in S \mid F(x) = \text{constant}\}$, then ∇F is a normal vector field of Σ . If Σ is spacelike, that is, Σ has a positive definite metric induced from (S, ds^2) , then ∇F forms a timelike normal vector field on Σ . Since

$$\begin{aligned}
\nabla F &= g^{tt}F_t\partial_t + g^{rr}F_r\partial_r + g^{\theta\theta}F_\theta\partial_\theta + g^{\phi\phi}F_\phi\partial_\phi \\
&= -\frac{1}{h(r)}F_t\partial_t + h(r)F_r\partial_r + \frac{1}{r^2}F_\theta\partial_\theta + \frac{1}{r^2\sin^2\theta}F_\phi\partial_\phi,
\end{aligned}$$

the spacelike condition on Σ is equivalent to

$$\langle \nabla F, \nabla F \rangle < 0 \Leftrightarrow -\frac{1}{h(r)}F_t^2 + h(r)F_r^2 + \frac{1}{r^2}F_\theta^2 + \frac{1}{r^2\sin^2\theta}F_\phi^2 < 0. \quad (4)$$

When Σ is a level set of F and is spacelike, we can without loss of generality assume that ∇F is future directed (or replace F by $-F$). That is,

$$N = \frac{\nabla F}{\sqrt{-\langle \nabla F, \nabla F \rangle}}$$

is future directed unit timelike normal vector field on Σ .

Let $\{e_i\}_{i=1}^3$ be a basis on Σ , then mean curvature of Σ is

$$H = \frac{1}{3} \sum_{i=1}^3 g^{ij} h_{ij} = \frac{1}{3} \sum_{i=1}^3 g^{ij} \langle \nabla_{e_i} N, e_j \rangle = \frac{1}{3} \sum_{i=1}^3 \frac{g^{ij}}{\sqrt{-\langle \nabla F, \nabla F \rangle}} \langle \nabla_{e_i}(\nabla F), e_j \rangle.$$

3.1 SS-CMC solutions in region I

We start to study SS-CMC solutions in the Schwarzschild exterior which maps to the region I in the Kruskal extension.

Proposition 2. *Suppose $\Sigma^1 = (f_1(r), r, \theta, \phi)$ is a SS-CMC hypersurface in the Schwarzschild exterior. Then the mean curvature equation is*

$$f_1'' + \left(\left(\frac{1}{h} - (f_1')^2 h \right) \left(\frac{2h}{r} + \frac{h'}{2} \right) + \frac{h'}{h} \right) f_1' - 3H \left(\frac{1}{h} - (f_1')^2 h \right)^{\frac{3}{2}} = 0,$$

where $h(r) = 1 - \frac{2M}{r}$ and H is the mean curvature. The explicit expression of f_1' can be derived as

$$f_1'(r; H, c_1) = \frac{l_1(r; H, c_1)}{h(r)} \sqrt{\frac{1}{1 + l_1^2(r; H, c_1)}}, \quad \text{where} \quad l_1(r; H, c_1) = \frac{1}{\sqrt{h(r)}} \left(Hr + \frac{c_1}{r^2} \right)$$

for some constant c_1 , and the integration gives

$$f_1(r; H, c_1, \bar{c}_1) = \int_{r_1}^r \frac{l_1(r; H, c_1)}{h(r)} \sqrt{\frac{1}{1 + l_1^2(r; H, c_1)}} dr + \bar{c}_1, \quad (5)$$

where \bar{c}_1 is a constant and $r_1 \in (2M, \infty)$ is fixed.

Proof. Take $F(t, r, \theta, \phi) = -t + f_1(r)$ and Σ^1 becomes a level set of F . In addition, $\nabla F = \frac{1}{h(r)} \partial_t + f_1'(r) h(r) \partial_r$ is future directed because it points to the direction of increasing t . The spacelike condition (4) is equivalent to

$$-\frac{1}{h(r)} + (f_1'(r))^2 h(r) < 0 \Leftrightarrow |f_1'(r) h(r)| < 1. \quad (6)$$

Thus the future directed unit timelike normal vector can be expressed as

$$e_4 = \frac{\left(\frac{1}{h(r)}, h(r) f_1'(r), 0, 0 \right)}{\sqrt{\frac{1}{h(r)} - (f_1'(r))^2 h(r)}}. \quad (7)$$

There is a canonical orthonormal frame on Σ^1

$$e_1 = \frac{(0, 0, 1, 0)}{r}, \quad e_2 = \frac{(0, 0, 0, 1)}{r \sin \theta}, \quad \text{and} \quad e_3 = \frac{(f_1'(r), 1, 0, 0)}{\sqrt{\frac{1}{h(r)} - (f_1'(r))^2 h(r)}}. \quad (8)$$

The second fundamental form of Σ^1 can be calculated directly, and we have

$$\begin{aligned} h_{11} &= \frac{1}{\left(\frac{1}{h} - (f_1')^2 h \right)^{\frac{1}{2}}} \frac{h f_1'}{r}, & h_{22} &= \frac{1}{\left(\frac{1}{h} - (f_1')^2 h \right)^{\frac{1}{2}}} \frac{h f_1'}{r}, \\ h_{33} &= \frac{1}{\left(\frac{1}{h} - (f_1')^2 h \right)^{\frac{1}{2}}} \left(\frac{1}{\frac{1}{h} - (f_1')^2 h} \left(f_1'' + \frac{h' f_1'}{h} \right) + \frac{h' f_1'}{2} \right), \end{aligned}$$

and $h_{ij} = 0$ for $i \neq j$. Hence the mean curvature equation becomes

$$f_1'' + \left(\left(\frac{1}{h} - (f_1')^2 h \right) \left(\frac{2h}{r} + \frac{h'}{2} \right) + \frac{h'}{h} \right) f_1' - 3H \left(\frac{1}{h} - (f_1')^2 h \right)^{\frac{3}{2}} = 0, \quad (9)$$

which is a second order ordinary differential equation.

To solve $f_1(r)$, we define $\sin(\eta(r)) = f_1'(r)h(r)$. The spacelike condition (6) implies that the change of variable is meaningful, and we can choose the range of η in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Equation (9) becomes

$$(\tan \eta)' + \left(\frac{2}{r} + \frac{h'}{2h} \right) \tan \eta - 3H \left(\frac{1}{h^{\frac{1}{2}}} \right) = 0 \Rightarrow \tan \eta = \frac{1}{\sqrt{h(r)}} \left(Hr + \frac{c_1}{r^2} \right),$$

where c_1 is a constant. We write $l_1(r; H, c_1) = \frac{1}{\sqrt{h(r)}} \left(Hr + \frac{c_1}{r^2} \right) = \tan \eta$ for convenience. On the other hand, since $\sin \eta = f_1' h$, it gives $\tan \eta = \frac{f_1' h}{\sqrt{1 - (f_1' h)^2}}$. Therefore,

$$\begin{aligned} \frac{f_1' h}{\sqrt{1 - (f_1' h)^2}} = l_1 &\Rightarrow f_1' = \frac{l_1}{h} \sqrt{\frac{1}{1 + l_1^2}} \quad \text{and} \\ f_1(r; H, c_1, \bar{c}_1) &= \int_{r_1}^r \frac{l_1(r; H, c_1)}{h(r)} \sqrt{\frac{1}{1 + l_1^2(r; H, c_1)}} dr + \bar{c}_1, \end{aligned}$$

where \bar{c}_1 is a constant and $r_1 \in (2M, \infty)$ is a fixed number. \square

Here are some remarks on the SS-CMC solutions in (5).

Remark 1. We can choose r_1 satisfying $r_1 + 2M \ln |r_1 - 2M| = 0$.

Remark 2. The sign of $l_1(r)$ is the same as the sign of $f_1'(r)$, and the condition for $l_1(r) \gtrless 0$ is equivalent to $Hr^3 + c_1 \gtrless 0$. So $f_1'(r)$ changes sign at most once. More explicitly, we have

- (a) If $H > 0$ and $c_1 \geq -8M^3H$, then $f_1(r)$ is increasing on $r > 2M$.
- (b) If $H > 0$ and $c_1 < -8M^3H$, then $f_1(r)$ is decreasing on $\left(2M, \left(\frac{-c_1}{H}\right)^{\frac{1}{3}}\right)$, and increasing on $\left(\left(\frac{-c_1}{H}\right)^{\frac{1}{3}}, \infty\right)$. Function $f_1(r)$ has a unique minimum at $r = \left(\frac{-c_1}{H}\right)^{\frac{1}{3}}$.
- (c) If $H < 0$ and $c_1 \leq -8M^3H$, then $f_1(r)$ is decreasing on $r > 2M$.
- (d) If $H < 0$ and $c_1 > -8M^3H$, then $f_1(r)$ is increasing on $\left(2M, \left(\frac{-c_1}{H}\right)^{\frac{1}{3}}\right)$, and decreasing on $\left(\left(\frac{-c_1}{H}\right)^{\frac{1}{3}}, \infty\right)$. Function $f_1(r)$ has a unique maximum at $r = \left(\frac{-c_1}{H}\right)^{\frac{1}{3}}$.

Remark 3. The second fundamental form of Σ^1 with basis (8) satisfies

$$h_{11} = h_{22} = H + \frac{c_1}{r^3}, \quad h_{33} = H - \frac{2c_1}{r^3}, \quad \text{and} \quad h_{11}, h_{22}, h_{33} \rightarrow H \quad \text{as} \quad r \rightarrow \infty.$$

In particular, if $c_1 = 0$, then $h_{11} = h_{22} = h_{33} = H$. We call this hypersurface *umbilical slice*.

Remark 4. The graphs of $f_1(r; H, c_1, \bar{c}_1)$ for $\bar{c}_1 \in \mathbb{R}$ gives a foliation in the Schwarzschild exterior.

3.2 Asymptotic behavior of SS-CMC solutions in region I

We analyze the asymptotic behavior of SS-CMC solutions $f_1(r)$ in this subsection. Here we omit the dependency of f_1 on H , c_1 , \bar{c}_1 when there is no confusion.

Proposition 3. *For a SS-CMC hypersurface $\Sigma^1 = (f_1(r), r, \theta, \phi)$, we have $\lim_{r \rightarrow \infty} f_1'(r) = 1$ if $H > 0$; $\lim_{r \rightarrow \infty} f_1'(r) = -1$ if $H < 0$, and $\lim_{r \rightarrow \infty} f_1'(r) = 0$ if $H = 0$. Furthermore, Σ^1 is asymptotically null for $H \neq 0$ as $r \rightarrow \infty$, and Σ^1 is asymptotically to some constant slice ($t = t_0, r, \theta, \phi$) for $H = 0$ as $r \rightarrow \infty$.*

Proof. Since

$$\lim_{r \rightarrow \infty} f_1'(r) = \lim_{r \rightarrow \infty} \frac{l_1}{h(r)} \sqrt{\frac{1}{1 + l_1^2}} = \lim_{r \rightarrow \infty} \frac{Hr + \frac{c_1}{r^2}}{\left(1 - \frac{2M}{r}\right) \sqrt{1 - \frac{2M}{r} + \left(Hr + \frac{c_1}{r^2}\right)^2}},$$

the limit is 0 if $H = 0$, and is $\frac{H}{|H|}$ if $H \neq 0$.

We compute

$$\langle \nabla F, \nabla F \rangle = -\frac{1}{h(r)} + h(r)(f_1'(r))^2 = \frac{-1}{h(r)(1 + l_1^2)} = \frac{-1}{h(r) + \left(Hr + \frac{c_1}{r^2}\right)^2}, \quad (10)$$

and have $\lim_{r \rightarrow \infty} \langle \nabla F, \nabla F \rangle = 0$ if $H \neq 0$. \square

Proposition 4. *For a SS-CMC hypersurface $\Sigma^1 = (f_1(r; H, c_1, \bar{c}_1), r, \theta, \phi)$ in the Schwarzschild exterior, the following conclusions hold:*

- (a) *If $c_1 < -8M^3H$, then $f_1'(r) < 0$ near $r = 2M$, and $f_1'(r)$ is of order $O((r - 2M)^{-1})$. It implies that $\lim_{r \rightarrow 2M^+} f_1(r) = \infty$.*
- (b) *If $c_1 = -8M^3H$, then $H \cdot f_1'(r) \geq 0$, and $f_1'(r)$ is of order $O((r - 2M)^{-\frac{1}{2}})$ when $H \neq 0$. It implies that $\lim_{r \rightarrow 2M^+} f_1(r)$ is finite.*
- (c) *If $c_1 > -8M^3H$, then $f_1'(r) > 0$ near $r = 2M$, and $f_1'(r)$ is of order $O((r - 2M)^{-1})$. It implies that $\lim_{r \rightarrow 2M^+} f_1(r) = -\infty$.*

When $c_1 \neq -8M^3H$, the curve $(f_1(r), r)$ in (t, r) spacetime is bounded by two null geodesics near $r = 2M$. For all $c_1 \in \mathbb{R}$, the spacelike condition is preserved as $r \rightarrow 2M^+$.

Proof. From (10), we know that if $c_1 \neq -8M^3H$, then $\lim_{r \rightarrow 2M^+} \langle \nabla F, \nabla F \rangle = \frac{-1}{\left(2MH + \frac{c_1}{4M^2}\right)^2} < 0$, and if $c_1 = -8M^3H$, then

$$\lim_{r \rightarrow 2M^+} \langle \nabla F, \nabla F \rangle = \lim_{r \rightarrow 2M^+} \frac{-1}{\frac{r-2M}{r} + \left(\frac{H(r-2M)(r^2+2Mr+4M^2)}{r^2}\right)^2} = -\infty.$$

Hence the spacelike condition is preserved as $r \rightarrow 2M^+$ for all $c_1 \in \mathbb{R}$.

Now we prove the asymptotic behavior of $f_1(r)$.

If $c_1 < -8M^3H$, then $f_1'(r) < 0$ (and thus $l_1(r) < 0$) on $r \in (2M, 2M + \delta_1)$ for some $\delta_1 > 0$.

Therefore, on $(2M, 2M + \delta_1)$ by the Taylor's theorem, we have

$$\begin{aligned} f_1' &= \frac{l_1}{h} \sqrt{\frac{1}{1+l_1^2}} = \frac{1}{-h} \sqrt{1 - \frac{1}{1+l_1^2}} \\ &\approx \frac{1}{-h} \left(1 - \frac{1}{2} \left(\frac{1}{1+l_1^2} \right) - \frac{1}{8} \left(\frac{1}{1+l_1^2} \right)^2 - \frac{3}{16} \left(\frac{1}{1+l_1^2} \right)^3 - \dots \right) \\ &\approx \frac{1}{-h} + \frac{1}{2} \frac{1}{\left(h + \left(Hr + \frac{c_1}{r^2} \right)^2 \right)} + \frac{1}{8} \frac{h}{\left(h + \left(Hr + \frac{c_1}{r^2} \right)^2 \right)^2} + \dots \\ &= \frac{1}{-h} + \text{remainder terms.} \end{aligned}$$

Remainder terms can be bounded above by $\frac{1}{2\left(Hr + \frac{c_1}{r^2}\right)^2}$, so $f_1'(r)$ is of order $O((r - 2M)^{-1})$.

Furthermore, we have

$$\frac{1}{-h(r)} \leq f_1'(r) \leq \frac{1}{-h(r)} + \frac{1}{2\left(Hr + \frac{c_1}{r^2}\right)^2} \quad (11)$$

on $(2M, 2M + \delta_1)$. We integrate inequalities (11) and get

$$\int_r^{r_1} -\frac{x}{x-2M} dx \leq \int_r^{r_1} f_1'(x) dx \leq \int_r^{r_1} \left(-\frac{x}{x-2M} + \frac{1}{2\left(Hx + \frac{c_1}{x^2}\right)^2} \right) dx.$$

The integral $\int_r^{r_1} \frac{1}{2\left(Hx + \frac{c_1}{x^2}\right)^2} dx$ is finite, and we denote it by C_1 . It follows that

$$\begin{aligned} &-(r_1 + 2M \ln(r_1 - 2M)) + (r + 2M \ln(r - 2M)) \\ &\leq f_1(r_1) - f_1(r) \\ &\leq -(r_1 + 2M \ln(r_1 - 2M)) + (r + 2M \ln(r - 2M)) + C_1 \\ \Rightarrow &-(r + 2M \ln(r - 2M)) + C_2 - C_1 \leq f_1(r) \leq -(r + 2M \ln(r - 2M)) + C_2, \end{aligned}$$

where $C_2 = f_1(r_1) + (r_1 + 2M \ln(r_1 - 2M))$. Hence the curve $t = f_1(r)$ is bounded by two null geodesics $t + (r + 2M \ln(r - 2M)) = C_2 - C_1$ and $t + (r + 2M \ln(r - 2M)) = C_2$ near $r = 2M$.

If $c_1 = -8M^3H$, then

$$l_1 = \left(\frac{r}{r-2M} \right)^{\frac{1}{2}} \left(\frac{Hr^3 - 8M^3H}{r^2} \right) = H \left(\frac{r-2M}{r} \right)^{\frac{1}{2}} \left(\frac{r^2 + 2Mr + 4M^2}{r} \right).$$

Direct computation gives

$$f_1' = H \left(\frac{r}{r-2M} \right)^{\frac{1}{2}} \left(\frac{r(r^2 + 2Mr + 4M^2)^2}{r^3 + H^2(r-2M)(r^2 + 2Mr + 4M^2)^2} \right)^{\frac{1}{2}},$$

and thus f_1' is of order $O((r-2M)^{-\frac{1}{2}})$ when $H \neq 0$.

If $c_1 > -8M^3H$, then both $f_1'(r)$ and $l_1(r)$ are positive on $(2M, 2M + \delta_2)$ for some $\delta_2 > 0$.

By the Taylor's theorem, we have

$$f_1' = \frac{1}{h} \sqrt{1 - \frac{1}{1+l_1^2}} \approx \frac{1}{h} - \frac{1}{2} \frac{1}{\left(h + \left(Hr + \frac{c_1}{r^2}\right)^2\right)} - \frac{1}{8} \frac{h}{\left(h + \left(Hr + \frac{c_1}{r^2}\right)^2\right)^2} - \dots.$$

The remainder terms are greater than $\frac{-1}{2\left(Hr + \frac{c_1}{r^2}\right)^2}$ on $(2M, 2M + \delta_2)$. This implies

$$\frac{1}{h(r)} - \frac{1}{2\left(Hr + \frac{c_1}{r^2}\right)^2} \leq f_1'(r) \leq \frac{1}{h(r)}.$$

We integrate the above inequalities and get

$$\int_r^{r_1} \left(\frac{1}{h(x)} - \frac{1}{2\left(Hx + \frac{c_1}{x^2}\right)^2} \right) dx \leq \int_r^{r_1} f_1'(x) dx \leq \int_r^{r_1} \frac{1}{h(x)} dx.$$

The integral $\int_r^{r_1} \frac{1}{2\left(Hx + \frac{c_1}{x^2}\right)^2} dx$ is finite, and we denote it by C_3 . It follows that

$$\begin{aligned} & r_1 + 2M \ln(r_1 - 2M) - (r + 2M \ln(r - 2M)) - C_3 \\ & \leq f_1(r_1) - f_1(r) \\ & \leq r_1 + 2M \ln(r_1 - 2M) - (r + 2M \ln(r - 2M)) \\ \Rightarrow & (r + 2M \ln(r - 2M)) + C_4 \leq f_1(r) \leq (r + 2M \ln(r - 2M)) + C_3 + C_4, \end{aligned}$$

where $C_4 = f_1(r_1) - (r_1 + 2M \ln(r_1 - 2M))$. Hence the curve $t = f_1(r)$ is bounded by two null geodesics $t - (r + 2M \ln(r - 2M)) = C_4$ and $t - (r + 2M \ln(r - 2M)) = C_3 + C_4$ near $r = 2M$. \square

Figure 5 pictures SS-CMC hypersurfaces in the Schwarzschild exterior and their images in region I of the Kruskal extension.

4 SS-CMC solutions in region II

We consider SS-CMC hypersurfaces in the Schwarzschild interior in this section.

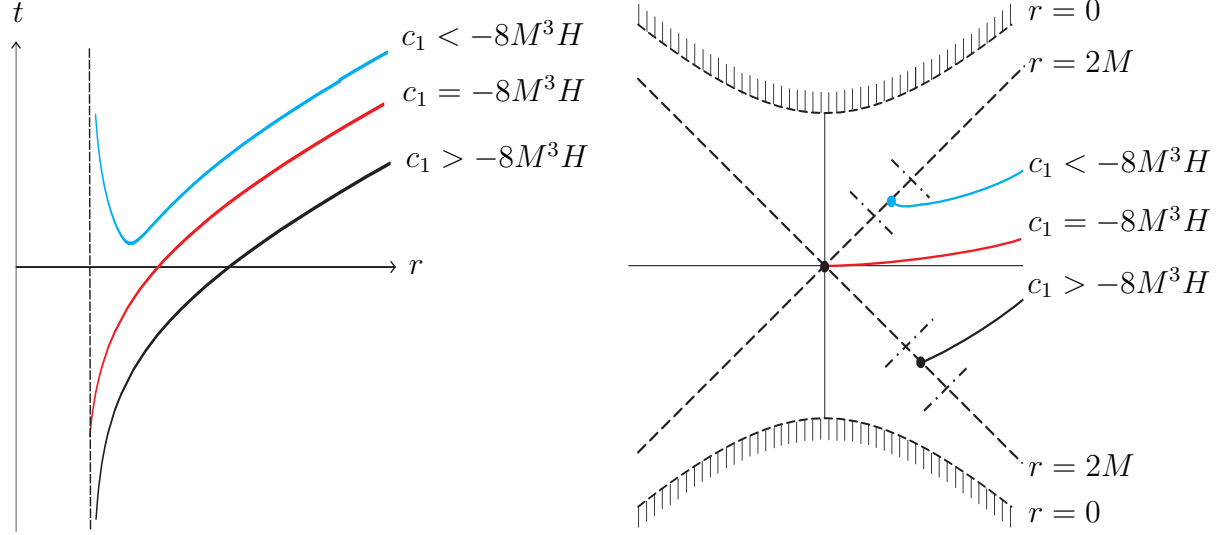


Figure 5: SS-CMC hypersurfaces in Schwarzschild exterior and region I.

4.1 Cylindrical hypersurfaces $r = \text{constant}$

Notice that $h(r) = 1 - \frac{2M}{r} < 0$ on $0 < r < 2M$, so in this region r -direction is timelike and t -direction is spacelike. Furthermore, $-\partial_r$ is future directed. We can assume that a SS-CMC hypersurface is written as $(t, g(t), \theta, \phi)$ for some function $r = g(t)$.

Proposition 5. [6] *Each constant slice $r = r_0, r_0 \in (0, 2M)$ is a SS-CMC hypersurface with*

$$H(r_0) = \frac{2r_0 - 3M}{3\sqrt{r_0^3(2M - r_0)}}.$$

These hypersurfaces are called cylindrical hypersurfaces.

Cylindrical hypersurfaces are known in [6]. Here we give a simple proof for completeness.

Proof. Choose $e_4 = (0, -\sqrt{-h(r)}, 0, 0)$ to be a future directed unit timelike normal vector, and there is a canonical orthonormal frame

$$e_1 = \frac{(0, 0, 1, 0)}{r}, \quad e_2 = \frac{(0, 0, 0, 1)}{r \sin \theta}, \quad e_3 = \frac{(1, 0, 0, 0)}{\sqrt{-h(r)}}$$

on constant slices. Since $\nabla_{\partial_t} \partial_r = \frac{h'(r)}{2h(r)} \partial_t$, $\nabla_V \partial_r = \frac{V}{r}$ for $V \in T_{(p,q)}\{p\} \times \mathbb{S}^2$, we have

$$\begin{aligned} h_{11} &= \langle \nabla_{e_1} e_4, e_1 \rangle = -\sqrt{-h(r)} \langle \nabla_{e_1} \partial_r, e_1 \rangle = -\frac{\sqrt{-h(r)}}{r}, \\ h_{22} &= \langle \nabla_{e_2} e_4, e_2 \rangle = -\sqrt{-h(r)} \langle \nabla_{e_2} \partial_r, e_2 \rangle = -\frac{\sqrt{-h(r)}}{r}, \\ h_{33} &= \langle \nabla_{e_3} e_4, e_3 \rangle = \frac{-1}{\sqrt{-h(r)}} \langle \nabla_{\partial_t} \partial_r, \partial_t \rangle = \frac{h'(r)}{2\sqrt{-h(r)}}, \end{aligned}$$

and $h_{ij} = 0$ for $i \neq j$. Hence the mean curvature is

$$H = \frac{1}{3} \left(\frac{-2\sqrt{-h(r)}}{r} + \frac{h'(r)}{2\sqrt{-h(r)}} \right) = \frac{1}{3\sqrt{-h(r)}} \left(\frac{2h(r)}{r} + \frac{h'(r)}{2} \right) = \frac{2r - 3M}{3\sqrt{r^3(2M - r)}},$$

which is a constant for each fixed $r \in (0, 2M)$. \square

The following corollary is an easy consequence of Proposition 5.

Corollary 1. *Cylindrical hypersurfaces $r = r_0, r_0 \in (0, 2M)$ have the following properties.*

- (a) *If $r_0 \in (0, \frac{3}{2}M)$, then $H(r_0) < 0$ and $\lim_{r \rightarrow 0^+} H(r) = -\infty$.*
- (b) *If $r_0 \in (\frac{3}{2}M, 2M)$, then $H(r_0) > 0$ and $\lim_{r \rightarrow 2M^-} H(r) = \infty$.*
- (c) *If $r_0 = \frac{3}{2}M$, then the cylindrical hypersurface is a maximal hypersurface.*

4.2 Noncylindrical SS-CMC hypersurfaces

For $r = g(t) \neq \text{constant}$, we consider its inverse function, and denote $t = f_2(r)$ whenever it is defined. Since $f_2(r)$ is obtained from the inverse function, we have $f_2'(r) \neq 0$ and will allow $f_2'(r) = \infty$ or $-\infty$.

Proposition 6. *Suppose $\Sigma^2 = (f_2(r), r, \theta, \phi)$ is a SS-CMC hypersurface in Schwarzschild interior. Then f_2' can be derived as*

$$f_2' = \begin{cases} \frac{1}{-h} \sqrt{\frac{l_2^2}{l_2^2 - 1}} & \text{if } f_2'(r) > 0 \\ \frac{1}{h} \sqrt{\frac{l_2^2}{l_2^2 - 1}} & \text{if } f_2'(r) < 0, \end{cases} \quad \text{where } l_2(r; H, c_2) = \frac{1}{\sqrt{-h(r)}} \left(-Hr - \frac{c_2}{r^2} \right).$$

The function l_2 should satisfy $l_2 > 1$, which implies $c_2 < 0$ when $H > 0$ and $c_2 < -8M^3H$ when $H < 0$. The integration of f_2' gives

$$f_2^*(r; H, c_2, \bar{c}_2) = \int_{r_2}^r \frac{1}{-h(r)} \sqrt{\frac{l_2^2(r; H, c_2)}{l_2^2(r; H, c_2) - 1}} dr + \bar{c}_2, \quad \text{or} \quad (12)$$

$$f_2^{**}(r; H, c_2, \bar{c}_2') = \int_{r_2'}^r \frac{1}{h(r)} \sqrt{\frac{l_2^2(r; H, c_2)}{l_2^2(r; H, c_2) - 1}} dr + \bar{c}_2' \quad (13)$$

according to the sign of $f_2'(r)$, where \bar{c}_2, \bar{c}_2' are constants, and r_2, r_2' are points in the domain of $f_2^*(r)$ and $f_2^{**}(r)$, respectively.

Remark 5. In this article, when we write $f_2(r)$, it means both $f_2^*(r)$ and $f_2^{**}(r)$.

Proof. First we consider the case $f'_2(r) > 0$. Denote $F(t, r, \theta, \phi) = -t + f_2(r)$, we have $\nabla F = \frac{1}{h(r)}\partial_t + f'_2(r)h(r)\partial_r$ is future directed because it is in the direction of decreasing r . The spacelike condition (4) is equivalent to

$$-\frac{1}{h(r)} + (f'_2(r))^2 h(r) < 0 \Leftrightarrow (f'_2(r)h(r))^2 > 1. \quad (14)$$

Hence future directed timelike normal vector is

$$e_4 = \frac{\left(\frac{1}{h(r)}, h(r)f'_2(r), 0, 0\right)}{\sqrt{\frac{1}{h(r)} - (f'_2(r))^2 h(r)}},$$

which has the same expression as (7), and we can take a canonical orthonormal frame on Σ^2 with the same expressions as (8). Therefore, the mean curvature equation will be

$$f''_2 + \left(\left(\frac{1}{h} - (f'_2)^2 h\right)\left(\frac{2h}{r} + \frac{h'}{2}\right) + \frac{h'}{h}\right) f'_2 - 3H \left(\frac{1}{h} - (f'_2)^2 h\right)^{\frac{3}{2}} = 0. \quad (15)$$

To solve $f_2(r)$, from (14) we can make change of variable by $\sec(\eta(r)) = f'_2(r)h(r)$. Since $h(r) = 1 - \frac{2M}{r} < 0$ on $0 < r < 2M$, we can choose the range of η to be $(\frac{\pi}{2}, \pi)$. Then equation (15) becomes

$$(\csc \eta)' + \left(\frac{2}{r} + \frac{h'}{2h}\right) \csc \eta + 3H \frac{1}{(-h)^{\frac{1}{2}}} = 0 \Rightarrow \csc \eta = \frac{1}{\sqrt{-h(r)}} \left(-Hr - \frac{c_2}{r^2}\right), \quad (16)$$

where c_2 is a constant. When writing $l_2(r; H, c_2) = \frac{1}{\sqrt{-h(r)}} \left(-Hr - \frac{c_2}{r^2}\right) = \csc \eta$, we have

$$f'_2 = \frac{\sec \eta}{h} = \frac{1}{-h\sqrt{1 - \frac{1}{\csc^2 \eta}}} = \frac{1}{-h} \sqrt{\frac{l_2^2}{l_2^2 - 1}}.$$

We remark that $l_2 = \csc \eta > 1$ because $\eta \in (\frac{\pi}{2}, \pi)$.

For the case $f'_2(r) < 0$, we choose $F(t, r, \theta, \phi) = t - f_2(r)$ such that $\nabla F = -\frac{1}{h(r)}\partial_t - f'_2(r)h(r)\partial_r$ is future directed. The spacelike condition is the same as (14), but future directed timelike normal vector is

$$e_4 = \frac{\left(-\frac{1}{h(r)}, -f'_2(r)h(r), 0, 0\right)}{\sqrt{\frac{1}{h(r)} - (f'_2(r))^2 h(r)}}.$$

There is a canonical orthonormal frame

$$e_1 = \frac{(0, 0, 1, 0)}{r}, \quad e_2 = \frac{(0, 0, 0, 1)}{r \sin \theta}, \quad \text{and} \quad e_3 = \frac{(-f'_2(r), -1, 0, 0)}{\sqrt{\frac{1}{h(r)} - (f'_2(r))^2 h(r)}}$$

on Σ^2 such that it has the same orientation as the case of $f'_2(r) > 0$. The second fundamental form of Σ^2 in (S, ds^2) are

$$\begin{aligned} h_{11} &= -\frac{1}{\left(\frac{1}{h} - (f'_2)^2 h\right)^{\frac{1}{2}}} \frac{h f'_2}{r}, & h_{22} &= -\frac{1}{\left(\frac{1}{h} - (f'_2)^2 h\right)^{\frac{1}{2}}} \frac{h f'_2}{r}, \\ h_{33} &= \frac{1}{\left(\frac{1}{h} - (f'_2)^2 h\right)^{\frac{1}{2}}} \left(\frac{-1}{\frac{1}{h} - (f'_2)^2 h} \left(f''_2 + \frac{h' f'_2}{h} \right) - \frac{h' f'_2}{2} \right), \end{aligned}$$

and $h_{ij} = 0$ for $i \neq j$. Hence the mean curvature equation becomes

$$f''_2 + \left(\left(\frac{1}{h} - (f'_2)^2 h \right) \left(\frac{2h}{r} + \frac{h'}{2} \right) + \frac{h'}{h} \right) f'_2 + 3H \left(\frac{1}{h} - (f'_2)^2 h \right)^{\frac{3}{2}} = 0. \quad (17)$$

From (14), we can change variable by $\sec(\eta(r)) = f'_2(r)h(r)$, and the range of η can be chosen as $(0, \frac{\pi}{2})$ because $h(r) = 1 - \frac{2M}{r} < 0$ on $0 < r < 2M$. Then (17) becomes

$$(\csc \eta)' + \left(\frac{2}{r} + \frac{h'}{2h} \right) \csc \eta + 3H \frac{1}{(-h)^{\frac{1}{2}}} = 0 \Rightarrow \csc \eta = \frac{1}{\sqrt{-h(r)}} \left(-Hr - \frac{c_2}{r^2} \right),$$

which has the same expression as (16). Set $l_2(r; H, c_2) = \frac{1}{\sqrt{-h(r)}} \left(-Hr - \frac{c_2}{r^2} \right) = \csc \eta$, then we have

$$f'_2(r; H, c_2) = \frac{1}{h(r)} \sqrt{\frac{l_2^2(r; H, c_2)}{l_2^2(r; H, c_2) - 1}}.$$

We remark that $l_2 = \csc \eta > 1$ because $\eta \in (0, \frac{\pi}{2})$. □

4.3 Domain of SS-CMC solutions in region II

The condition $l_2(r) > 1$ will put restrictions on the domain of $f_2(r)$. We have

$$l_2(r) = \frac{1}{\sqrt{-h(r)}} \left(-Hr - \frac{c_2}{r^2} \right) > 1 \Rightarrow -Hr^3 - r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}} > c_2.$$

Define a function $k_H(r)$ on $(0, 2M)$ by

$$k_H(r) = -Hr^3 - r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}}, \quad (18)$$

then the domain of $f_2(r)$ will be

$$\{r \in (0, 2M) | k_H(r) > c_2\} \cup \{r \in (0, 2M) | k_H(r) = c_2 \text{ and } f_2(r) \text{ is finite}\}.$$

Now we analyze the function $k_H(r)$ to determine the set.

Proposition 7. *Consider $k_H(r)$ as in (18), then $k_H(r)$ has a unique minimum point at $r = r_H$, where r_H is determined by $3Hr_H^{\frac{3}{2}}(2M - r_H)^{\frac{1}{2}} = 2r_H - 3M$.*

Proof. We differentiate $k_H(r)$ to get

$$k'_H(r) = \frac{-r^{\frac{1}{2}}}{(2M-r)^{\frac{1}{2}}} \left(3Hr^{\frac{3}{2}}(2M-r)^{\frac{1}{2}} + 3M - 2r \right). \quad (19)$$

Denote $\bar{k}_H(r) = 3Hr^{\frac{3}{2}}(2M-r)^{\frac{1}{2}}$, then

$$\bar{k}'_H(r) = \frac{3Hr^{\frac{1}{2}}}{(2M-r)^{\frac{1}{2}}} (3M - 2r).$$

It implies $\bar{k}'_H(\frac{3}{2}M) = 0$, and $\bar{k}_H(r)$ is monotone on $(0, \frac{3}{2}M)$ and $(\frac{3}{2}M, 2M)$. Furthermore, $\bar{k}_H(r)$ and the function $p(r) = 2r - 3M$ intersect at $r = r_H$. (See Figure 6.) That is, $3Hr_H^{\frac{3}{2}}(2M - r_H)^{\frac{1}{2}} = 2r_H - 3M$ and $k'_H(r_H) = 0$, so r_H is the critical point of $k_H(r)$.

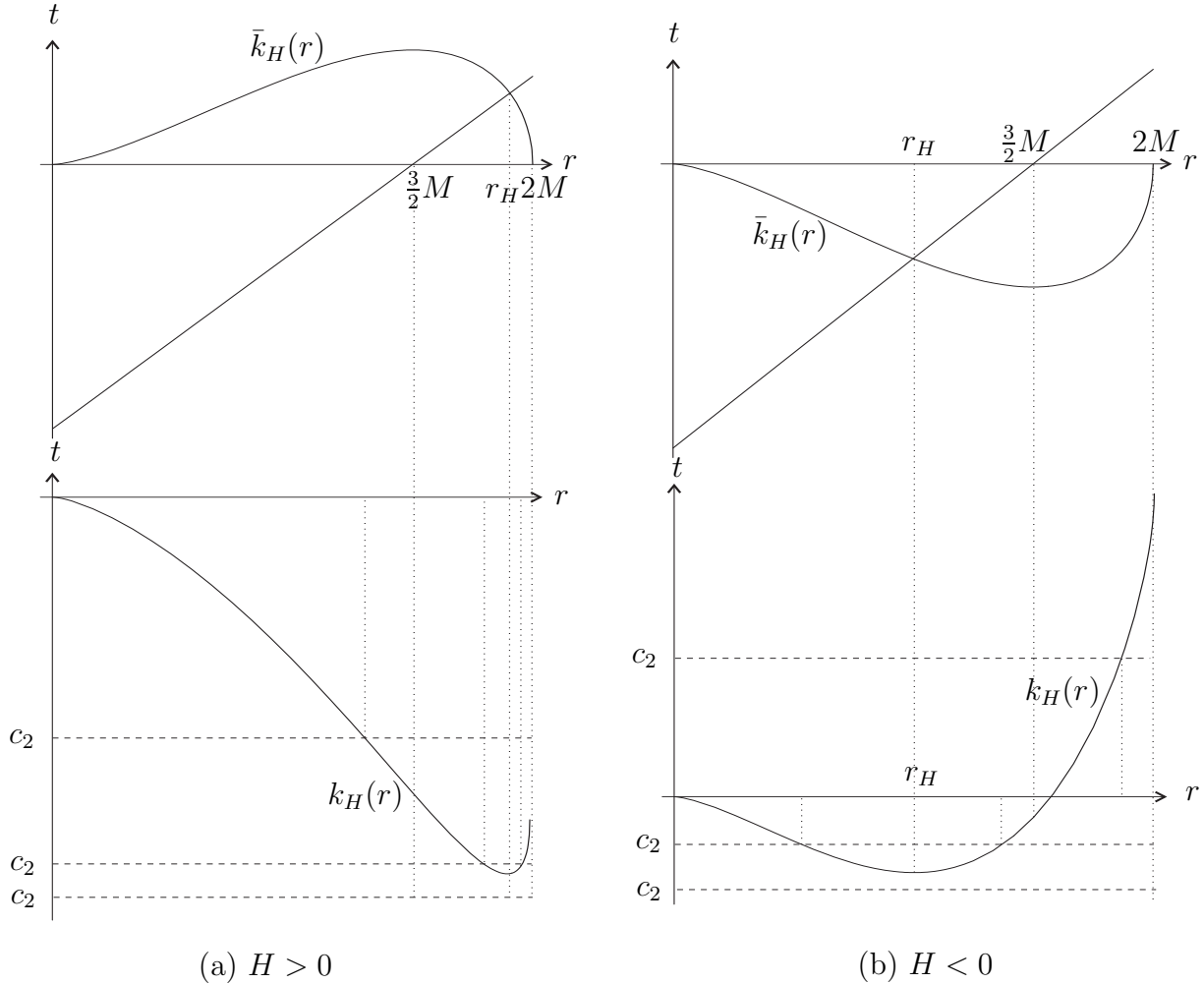


Figure 6: Graphs of $\bar{k}_H(r)$, $p(r) = 2r - 3M$, $k_H(r)$, and horizontal lines $l(r) = c_2$.

□

Proposition 8. Denote $c_H = \min_{r \in (0, 2M)} k_H(r) = k_H(r_H)$, where $k_H(r)$ is as in (18), and r_H is as in Proposition 7. There are three types of noncylindrical SS-CMC hypersurfaces $\Sigma^2 = (f_2(r), r, \theta, \phi)$ according to the value of c_2 , where $f_2(r) = f_2^*(r; H, c_2, \bar{c}_2)$ or $f_2^{**}(r; H, c_2, \bar{c}_2')$.

- (a) If $c_2 < c_H$, then $f_2(r)$ is defined on $(0, 2M)$.
- (b) If $c_2 = c_H$, then $f_2(r)$ is defined on $(0, r_H) \cup (r_H, 2M)$.
- (c) If $c_H < c_2 < \max(0, -8M^3H)$, then $f_2(r)$ is defined on $(0, r']$ or $[r'', 2M)$ for some r' and r'' , which depend on H and c_2 . When we take $r_2 = r'_2 = r'$ (or r'') and $\bar{c}_2 = \bar{c}_2'$ in (12) and (13), $\Sigma^2 = (f_2^*(r; H, c_2, \bar{c}_2) \cup f_2^{**}(r; H, c_2, \bar{c}_2'), r, \theta, \phi)$ is a complete SS-CMC hypersurface in the Schwarzschild interior.

Proof.

- (a) If $c_2 < c_H$, then $l_2(r) > 1$ for all $r \in (0, 2M)$, which implies $f_2(r)$ is defined on $(0, 2M)$.
- (b) If $c_2 = c_H$, then $f_2(r)$ is defined on $(0, r_H) \cup (r_H, 2M)$. We need to check the behavior of $f_2(r)$ as $r \rightarrow r_H$. First, we know $\lim_{r \rightarrow r_H} f_2'(r) = \infty$ or $-\infty$ because $l_2(r) = 1$. Next, noting that $c_2 = -Hr_H^3 - r_H^{\frac{3}{2}}(2M - r_H)^{\frac{1}{2}}$ and

$$f_2'(r) = \frac{-Hr^3 - c_2}{h(r)\sqrt{(-Hr^3 - c_2)^2 + r^3(r - 2M)}} \quad \text{or} \quad \frac{-Hr^3 - c_2}{-h(r)\sqrt{(-Hr^3 - c_2)^2 + r^3(r - 2M)}},$$

we expand $(-Hr^3 - c_2)^2 + r^3(r - 2M)$ in the power of $(r - r_H)$ to attain

$$\begin{aligned} & \sqrt{(-Hr^3 - c_2)^2 + r^3(r - 2M)} \\ &= \sqrt{P_1(r; r_H)(r - r_H)^2 + 2r_H^2 \left(-3Hr_H^{\frac{3}{2}}(2M - r_H)^{\frac{1}{2}} + 2r_H - 3M \right) (r - r_H)} \\ &= \sqrt{P_1(r; r_H)(r - r_H)^2}, \end{aligned}$$

where $P_1(r; r_H)$ is a polynomial. The last equality holds because r_H is the critical point of $k_H(r)$. Thus $f_2'(r) \sim O((r - r_H)^{-1})$ and $\lim_{r \rightarrow r_H} f_2(r) = \infty$ or $-\infty$. That is, the domain of $f_2(r)$ is $(0, r_H) \cup (r_H, 2M)$.

- (c) If $c_H < c_2 < \max(0, -8M^3H)$, then $f_2(r)$ is defined on $(0, r']$ or $(r'', 2M)$. Here we only discuss the case at r' because the case at r'' is similar. First, we know $\lim_{r \rightarrow r'} f_2'(r) = \infty$ or $-\infty$. Next, since $c_2 = -H(r')^3 - (r')^{\frac{3}{2}}(2M - r')^{\frac{1}{2}}$ is not a critical value of $k_H(r)$, the expansion of $(-Hr^3 - c_2)^2 + r^3(r - 2M)$ in the power of $(r - r')$ becomes

$$\begin{aligned} & \sqrt{(-Hr^3 - c_2)^2 + r^3(r - 2M)} \\ &= \sqrt{P_2(r; r')(r - r')^2 + 2(r')^2 \left(-3H(r')^{\frac{3}{2}}(2M - r')^{\frac{1}{2}} + 2r' - 3M \right) (r - r')}, \end{aligned}$$

where $P_2(r; r')$ is a polynomial, and $-3H(r')^{\frac{3}{2}}(2M - r')^{\frac{1}{2}} + 2r' - 3M \neq 0$. It implies $f_2'(r) \sim O((r - r')^{-\frac{1}{2}})$, and $\lim_{r \rightarrow r'} f_2(r)$ is a finite value. Domain of $f_2(r)$ can be extended to $r = r'$. When taking $r_2 = r'_2 = r'$ and $\bar{c}_2 = \bar{c}'_2$, we have $f_2^*(r'; H, c_2, \bar{c}_2) = f_2^{**}(r'; H, c_2, \bar{c}'_2) = \bar{c}_2$ and $\Sigma^2 = (f_2^*(r; H, c_2, \bar{c}_2) \cup f_2^{**}(r; H, c_2, \bar{c}'_2), r, \theta, \phi)$ is a complete SS-CMC hypersurface in the Schwarzschild interior.

□

Proposition 9. *In case (c) of Proposition 8, the SS-CMC hypersurface Σ^2 is C^∞ .*

Proof. It suffices to check the smoothness of Σ^2 at the joint point, and here we show the case of $r_2 = r'_2 = r'$. The case of $r_2 = r'_2 = r''$ is similar. Noting that $\bar{c}_2 = \bar{c}'_2$ and $r < r'$, we have $f^*(r) \leq \bar{c}_2$, $f^{**}(r) \geq \bar{c}_2$, and $f^*(r') = f^{**}(r') = \bar{c}_2$. Hence when rewrite the surface as a graph of $r = g(t)$, we have $g(\bar{c}_2) = r'$ and its inverse corresponds to $t = f_2^*(r)$ for $t \leq \bar{c}_2$ and to $t = f_2^{**}(r)$ for $t \geq \bar{c}_2$. Direct computation gives

$$g^{(2k+1)}(t) = \begin{cases} \sum_{i=0}^k A_{k,i} (l_2^2 - 1)^{i+\frac{1}{2}} & \text{if } t < \bar{c}_2 \\ (-1)^{2k+1} \sum_{i=0}^k A_{k,i} (l_2^2 - 1)^{i+\frac{1}{2}} & \text{if } t > \bar{c}_2, \end{cases}$$

$$g^{(2k)}(t) = \begin{cases} \sum_{i=0}^k B_{k,i} (l_2^2 - 1)^i & \text{if } t < \bar{c}_2 \\ (-1)^{2k} \sum_{i=0}^k B_{k,i} (l_2^2 - 1)^i & \text{if } t > \bar{c}_2, \end{cases}$$

where $A_{k,i}$ and $B_{k,i}$ are functions of h, l_2 and their derivatives with respect to r . As $t \rightarrow \bar{c}_2$, we have $r \rightarrow r'$ and $\lim_{r \rightarrow r'} l_2^2 - 1 = 0$, it implies that

$$\lim_{t \rightarrow \bar{c}_2^-} g^{(2k+1)}(t) = \lim_{t \rightarrow \bar{c}_2^+} g^{(2k+1)}(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \bar{c}_2^-} g^{(2k)}(t) = \lim_{t \rightarrow \bar{c}_2^+} g^{(2k)}(t) = B_{k,0}.$$

Hence Σ^2 is smooth.

□

4.4 Asymptotic behavior of SS-CMC solutions in region II

Next, we discuss the asymptotic behavior of SS-CMC hypersurfaces in Schwarzschild interior that will be needed in section 6.

Proposition 10. *For a SS-CMC hypersurface $\Sigma^2 = (f_2(r; H, c_2, \bar{c}_2), r, \theta, \phi)$ in Schwarzschild interior with $c_2 < -8M^3H$, we have $f_2'(r)$ is of order $O((2M - r)^{-1})$ as $r \rightarrow 2M^-$. It implies that $\lim_{r \rightarrow 2M^-} f_2(r) = \infty$ or $-\infty$, and the curve $(f_2(r), r)$ in (t, r) plane is bounded by two null geodesics as $r \rightarrow 2M^-$. Furthermore, the spacelike condition is preserved as $r \rightarrow 2M^-$.*

Proof. Since $c_2 < -8M^3H$, $f_2(r)$ is defined on $(2M - \delta_3, 2M)$ for some $\delta_3 > 0$. We only need to consider the case $f_2'(r) > 0$ because of symmetry. On one hand, since

$$f_2'(r) = \frac{1}{(-h)\sqrt{1 - \frac{r^3(2M-r)}{(-Hr^3-c_2)^2}}} \geq \frac{1}{-h},$$

we have

$$\begin{aligned}
\int_{r_2}^r f_2'(x) dx &\geq -(x + 2M \ln(2M - x)) \Big|_{x=r_2}^{x=r} \\
\Rightarrow f_2(r) &\geq f_2(r_2) + (r_2 + 2M \ln(2M - r_2)) - (r + 2M \ln(2M - r)) \\
&= -(r + 2M \ln(2M - r)) + C_5,
\end{aligned}$$

where $C_5 = f_2(r_2) + r_2 + 2M \ln(2M - r_2)$. The curve $t = f_2(r)$ is bounded below by the null geodesic $t + (r + 2M \ln(2M - r)) = C_5$ near $r = 2M$.

On the other hand, because $\frac{1}{l_2^2}$ is very small near $r = 2M$, by Taylor's expansion we get

$$\begin{aligned}
\sqrt{1 - \frac{1}{l_2^2}} &\approx 1 - \frac{1}{2} \left(\frac{1}{l_2^2} \right) - \frac{1}{8} \left(\frac{1}{l_2^2} \right)^2 - \dots \geq 1 - \left(\frac{1}{l_2^2} \right) - \left(\frac{1}{l_2^2} \right)^2 - \dots \\
&= 1 - \frac{1}{l_2^2 - 1} = 1 - \frac{-h}{(-Hr - \frac{c_2}{r^2})^2 - (-h)}
\end{aligned}$$

There is a constant $C_6 > 0$ such that $C_6 \left((-Hr - \frac{c_2}{r^2})^2 - 2(-h) \right) > 1$ on $(2M - \delta_4, 2M)$, a subset of $(2M - \delta_3, 2M)$. That is, we have

$$\frac{1}{(-Hr - \frac{c_2}{r^2})^2 - (-h)} < \frac{C_6}{1 + C_6(-h)}$$

and

$$\sqrt{1 - \frac{1}{l_2^2}} \geq 1 - \frac{C_6(-h)}{1 + C_6(-h)} = \frac{1}{1 + C_6(-h)}.$$

Thus

$$f_2'(r) = \frac{1}{(-h)\sqrt{1 - \frac{1}{l_2^2}}} \leq \frac{1}{(-h)}(1 + C_6(-h)) = \frac{1}{-h} + C_6,$$

which integrates to

$$\int_{r_2}^r f_2'(x) dx \leq \int_{r_2}^r \left(\frac{1}{-h(x)} + C_6 \right) dx.$$

Hence

$$\begin{aligned}
f_2(r) &\leq f_2(r_2) - (x + 2M \ln(2M - r)) \Big|_{x=r_2}^{x=r} + C_6(r - r_2) \\
&= -(r + 2M \ln(2M - r)) + C_5 + C_7,
\end{aligned}$$

where $C_5 = f_2(r_2) + (r_2 + 2M \ln(2M - r_2))$ and $C_7 = C_6(2M - r_2)$. The curve $t = f_2(r)$ is bounded above by the null geodesic $t + (r + 2M \ln(2M - r)) = C_5 + C_7$ near $r = 2M$.

Spacelike property can be extended at $r = 2M$ because

$$\lim_{r \rightarrow 2M^-} \langle \nabla F, \nabla F \rangle = \lim_{r \rightarrow 2M^-} \frac{-1}{(-Hr - \frac{c_2}{r^2})^2 + h(r)} = \frac{-1}{(-2MH - \frac{c_2}{4M^2})^2} < 0.$$

□

Figure 7 pictures SS-CMC hypersurfaces in Schwarzschild interior and their images in region Π of Kruskal extension.

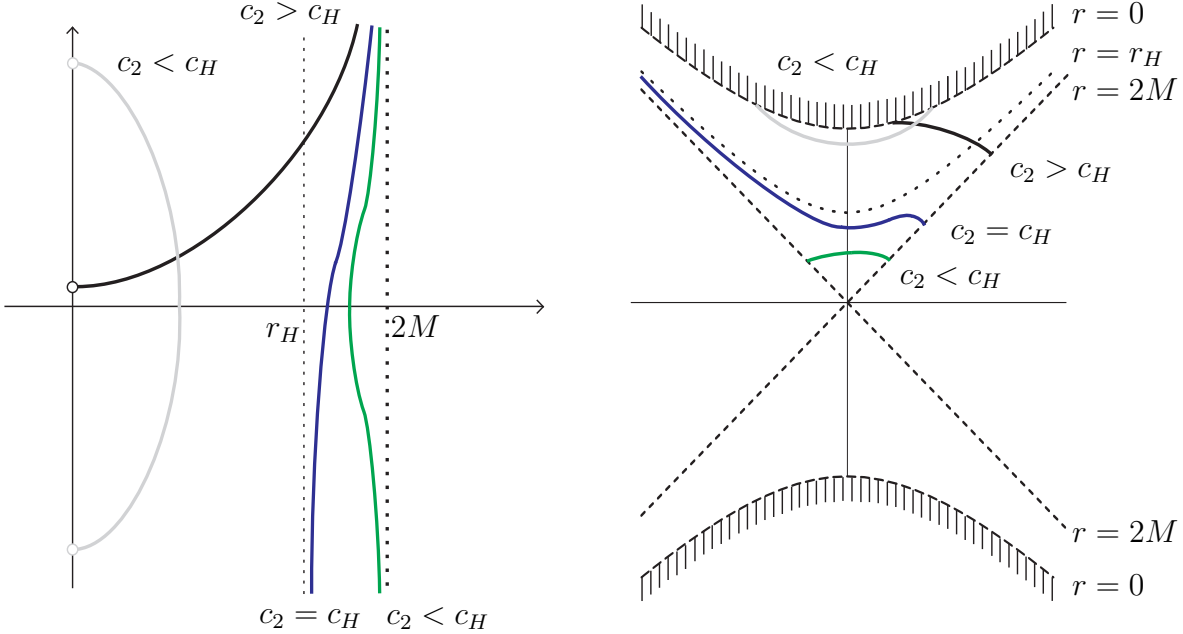


Figure 7: SS-CMC hypersurfaces in Schwarzschild interior and region Π .

5 SS-CMC Solutions in region I' and Π'

When the Schwarzschild exterior and interior map to region I and Π in the Kruskal extension, future directed timelike directions are directions of increasing t and decreasing r , respectively. However, when the Schwarzschild exterior and interior map to region I' and Π' , future directed timelike directions are directions of decreasing t and increasing r , respectively. Therefore, we need to modify the discussions in section 3 and 4 according to these differences for the SS-CMC solutions in region I' and Π' .

The constant mean curvature equation of a SS-CMC hypersurface $\Sigma^3 = (f_3(r), r, \theta, \phi)$ which maps to region I' of the Kruskal extension is

$$f_3'' + \left(\left(\frac{1}{h} - (f_3')^2 h \right) \left(\frac{2h}{r} + \frac{h'}{2} \right) + \frac{h'}{h} \right) f_3' + 3H \left(\frac{1}{h} - (f_3')^2 h \right)^{\frac{3}{2}} = 0. \quad (20)$$

Solutions to the equation (20) would be

$$f_3'(r; H, c_3) = \frac{l_3(r; H, c_3)}{h(r)} \sqrt{\frac{1}{1 + l_3^2(r; H, c_3)}}, \quad \text{where} \quad l_3(r; H, c_3) = \frac{1}{\sqrt{h(r)}} \left(-Hr - \frac{c_3}{r^2} \right),$$

and

$$f_3(r; H, c_3, \bar{c}_3) = \int_{r_3}^r \frac{l_3(r; H, c_3)}{h(r)} \sqrt{\frac{1}{1 + l_3^2(r; H, c_3)}} dr + \bar{c}_3, \quad (21)$$

where c_3 and \bar{c}_3 are constants, and $r_3 \in (2M, \infty)$ is fixed. We remark that for given constant mean curvature H , if $c_1 = c_3$, then $f_1'(r) = -f_3'(r)$.

Similarly, when we consider SS-CMC hypersurfaces in another Schwarzschild interior that maps to region Π' of the Kruskal extension, each constant slice $r = r_0, r_0 \in (0, 2M)$ are SS-CMC solutions.

Moreover, given a SS-CMC hypersurfaces $\Sigma^4 = (f_4(r), r, \theta, \phi)$ which maps to region Π' , the constant mean curvature equation of $f_4(r)$ is

$$\begin{cases} f_4'' + \left(\left(\frac{1}{h} - (f_4')^2 h \right) \left(\frac{2h}{r} + \frac{h'}{2} \right) + \frac{h'}{h} \right) f_4' + 3H \left(\frac{1}{h} - (f_4')^2 h \right)^{\frac{3}{2}} = 0 & \text{if } f_4'(r) > 0 \\ f_4'' + \left(\left(\frac{1}{h} - (f_4')^2 h \right) \left(\frac{2h}{r} + \frac{h'}{2} \right) + \frac{h'}{h} \right) f_4' - 3H \left(\frac{1}{h} - (f_4')^2 h \right)^{\frac{3}{2}} = 0 & \text{if } f_4'(r) < 0. \end{cases} \quad (22)$$

We can solve (22) to get

$$f_4'(r) = \begin{cases} \frac{1}{-h} \sqrt{\frac{l_4^2}{l_4^2 - 1}}, & \text{if } f_4'(r) > 0 \\ \frac{1}{h} \sqrt{\frac{l_4^2}{l_4^2 - 1}}, & \text{if } f_4'(r) < 0, \end{cases} \quad \text{where} \quad l_4(r; H, c_4) = \frac{1}{\sqrt{-h(r)}} \left(Hr + \frac{c_4}{r^2} \right).$$

The integration of $f_4'(r)$ gives

$$f_4^*(r; H, c_4, \bar{c}_4) = \int_{r_4}^r \frac{1}{-h(r)} \sqrt{\frac{l_4^2(r; H, c_4)}{l_4^2(r; H, c_4) - 1}} dr + \bar{c}_4, \quad \text{or} \quad (23)$$

$$f_4^{**}(r; H, c_4, \bar{c}_4') = \int_{r_4'}^r \frac{1}{h(r)} \sqrt{\frac{l_4^2(r; H, c_4)}{l_4^2(r; H, c_4) - 1}} dr + \bar{c}_4' \quad (24)$$

according to the sign of $f_4'(r)$, where $c_4, \bar{c}_4, \bar{c}_4'$ are constants, and r_4, r_4' are fixed numbers in the domain of $f_4^*(r)$ and $f_4^{**}(r)$, respectively. The function $l_4(r)$ should satisfy $l_4(r) > 1$, which implies $c_4 > -8M^3H$ when $H > 0$ and $c_4 > 0$ when $H < 0$. In addition, we allow $f_4'(r) = \pm\infty$ at some point.

In this article, when we write $f_4(r)$, it means both $f_4^*(r)$ and $f_4^{**}(r)$.

The condition $l_4(r) > 1$ will put restrictions on the domain of $f_4(r)$. We have

$$l_4(r) = \frac{1}{\sqrt{-h(r)}} \left(Hr + \frac{c_4}{r^2} \right) > 1 \Rightarrow -Hr^3 + r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}} < c_4.$$

Define

$$\tilde{k}_H(r) = -Hr^3 + r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}}. \quad (25)$$

Then the domain of $f_4(r)$ will be

$$\{r \in (0, 2M) | \tilde{k}_H(r) < c_4\} \cup \{r \in (0, 2M) | \tilde{k}_H(r) = c_4 \text{ and } f_4(r) \text{ is finite}\}.$$

By similar arguments as in Proposition 7, we can analyze $\tilde{k}_H(r)$ and illustrate its graph according to the sign of H in Figure 8. Our conclusion is

Proposition 11. *Consider $\tilde{k}_H(r)$ as in (25), then $\tilde{k}_H(r)$ has a unique maximum point at $r = R_H$, where R_H is determined by $-3HR_H^{\frac{3}{2}}(2M - R_H)^{\frac{1}{2}} = 2R_H - 3M$.*

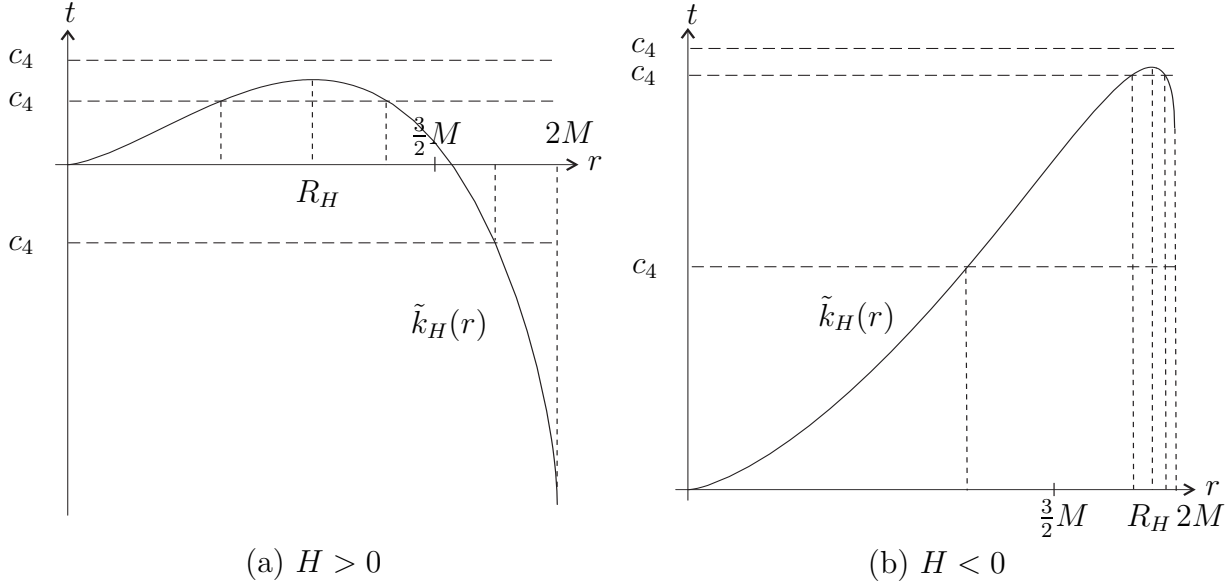


Figure 8: Graphs of $\tilde{k}_H(r)$ and horizontal lines $l(r) = c_4$.

Corresponding to Proposition 8 and Proposition 9, the following results can be proved by the same method.

Proposition 12. *Denote $C_H = \max_{r \in (0, 2M)} \tilde{k}_H(r) = \tilde{k}_H(R_H)$, where $\tilde{k}_H(r)$ is as in (25), and R_H is as in Proposition 11. There are three types of noncylindrical SS-CMC hypersurfaces $\Sigma^4 = (f_4(r), r, \theta, \phi)$ according to the value of c_4 , where $f_4(r) = f_4^*(r; H, c_4, \bar{c}_4), r, \theta, \phi$ or $(f_4^{**}(r; H, c_4, \bar{c}_4'), r, \theta, \phi)$.*

- (a) If $c_4 > C_H$, then $f_4(r)$ is defined on $(0, 2M)$.
- (b) If $c_4 = C_H$, then $f_4(r)$ is defined on $(0, R_H) \cup (R_H, 2M)$.
- (c) If $\min(0, -8M^3H) < c_4 < C_H$, then $f_4(r)$ is defined on $(0, r']$ or $[r'', 2M)$ for some r' and r'' , which depend on H and c_4 . When we take $r_4 = r'_4 = r'$ (or r'') and $\bar{c}_4 = \bar{c}'_4$ in (23) and (24), $\Sigma^4 = (f_4^*(r; H, c_4, \bar{c}_4) \cup f_4^{**}(r; H, c_4, \bar{c}'_4), r, \theta, \phi)$ is a complete SS-CMC hypersurface in the Schwarzschild interior.

Remark 6. For given constant mean curvature H , if $c_2 = c_4$, then $f'_2(r) = f'_4(r)$.

Proposition 13. In case (c) of Proposition 12, the SS-CMC hypersurface Σ^4 is C^∞ .

6 Complete and smooth SS-CMC hypersurfaces

In this section we will investigate how to join solutions from different regions at $r = 2M$ to construct complete hypersurfaces in the Kruskal extension, and discuss the smoothness property at each joint point. First, we discuss SS-CMC hypersurfaces in region I, II, and I'. As in section 4, denote $c_H = \min_{r \in (0, 2M)} -Hr^3 - r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}}$, and notice that $c_H < -8M^3H$ from Figure 6, then we have the following theorems.

Theorem 1. Given constant mean curvature H , $c_1 < -8M^3H$, and $\bar{c}_1 \in \mathbb{R}$, it determines a SS-CMC hypersurface $\Sigma^1_{H, c_1, \bar{c}_1}$ in region I. This $\Sigma^1_{H, c_1, \bar{c}_1}$ connects smoothly with a SS-CMC hypersurface $\Sigma^2_{H, c_1, \bar{c}_2}$ for some \bar{c}_2 in region II. Moreover,

- (a) when $c_1 < c_H$, the corresponding SS-CMC hypersurface $\Sigma^2_{H, c_1, \bar{c}_2}$ in Schwarzschild is defined on $(0, 2M)$, and $\Sigma^1 \cup \Sigma^2$ forms a complete and smooth SS-CMC hypersurface in the Kruskal extension with two ends.
- (b) when $c_1 = c_H$, the corresponding SS-CMC hypersurface $\Sigma^2_{H, c_1, \bar{c}_2}$ in Schwarzschild is defined on $(r_H, 2M)$, and $\Sigma^1 \cup \Sigma^2$ forms a complete and smooth SS-CMC hypersurface in the Kruskal extension with two ends. Here r_H is defined as in Proposition 7.
- (c) when $c_H < c_1 < -8M^3H$, $\Sigma^2_{H, c_1, \bar{c}_2}$ connects smoothly to a SS-CMC hypersurface $\Sigma^3_{H, c_1, \bar{c}_3}$ for some \bar{c}_3 in region I'. Then $\Sigma^1 \cup \Sigma^2 \cup \Sigma^3$ forms a complete and smooth SS-CMC hypersurface in the Kruskal extension with two ends.

Remark 7. The followings are some descriptions for the ends in each case of Theorem 1.

- In case (a), among the two ends, one is toward the space infinity $r = \infty$ in the first Schwarzschild exterior, and the other is toward the space singularity $r = 0$ in the first Schwarzschild interior.

- In case (b), one end is toward the space infinity in the first Schwarzschild exterior, and the other end is asymptotic to a cylindrical SS-CMC hypersurface $r = r_H$ in the first Schwarzschild interior.
- In case (c), the two ends are toward space infinities $r = \infty$ in the first and second Schwarzschild exteriors, respectively.

Relations between \bar{c}_1 , \bar{c}_2 , and \bar{c}_3 are described as below.

Theorem 2. Consider $c_1 < -8M^3H$ and complete SS-CMC hypersurfaces as in Theorem 1. Take $r_1 > 2M$ satisfying $r_1 + 2M \ln |r_1 - 2M| = 0$ and denote $\bar{f}'(r) = f'_1(r) + \frac{1}{h(r)}$, which can be expressed as

$$\bar{f}'(r) = \frac{r^4}{(Hr^3 + c_1)^2 + r^3(r - 2M) - (Hr^3 + c_1)\sqrt{(Hr^3 + c_1)^2 + r^3(r - 2M)}}. \quad (26)$$

Then

$$\bar{c}_2 = \bar{c}_1 - \int_{r_2}^{r_1} \bar{f}'(r) dr - (r_2 + 2M \ln |r_2 - 2M|),$$

where $r_2 < 2M$ is in the domain of $f_2^*(r)$. In case (c) of Theorem 1, if we take $r_2 = r'_2 = r''$, $\bar{c}'_2 = \bar{c}_2$, and $r_3 = r_1$ in (12) and (13), then

$$\begin{aligned} \bar{c}_3 &= \bar{c}_2 - \int_{r''}^{r_3} \bar{f}'(r) dr - (r'' + 2M \ln |r'' - 2M|) \\ &= \bar{c}_1 - 2 \int_{r''}^{r_1} \bar{f}'(r) dr - 2(r'' + 2M \ln |r'' - 2M|). \end{aligned}$$

Proof of Theorem 1 and 2. First, we prove that the necessary condition for $\Sigma_{H,c_1,\bar{c}_1}^1$ and $\Sigma_{H,c_2,\bar{c}_2}^2$ (or $\Sigma_{H,c_2,\bar{c}_2}^2$ and $\Sigma_{H,c_3,\bar{c}_3}^3$) to join smoothly is $c_1 = c_2$ (or $c_2 = c_3$).

Given $\Sigma_{H,c_1,\bar{c}_1}^1 = (f_1(r; H, c_1, \bar{c}_1), r, \theta, \phi)$, by Proposition 4 we know that

$$f'_1(r) = \frac{-1}{h} \sqrt{\frac{l_1^2}{1 + l_1^2}} = -\frac{1}{h(r)} + \text{finite term} \quad \text{near } r = 2M.$$

The limit of the finite term of $f'_1(r)$ at $r = 2M$ is

$$\lim_{r \rightarrow 2M^+} \left(\frac{-1}{h} \sqrt{\frac{l_1^2}{1 + l_1^2}} + \frac{1}{h} \right) = \frac{1}{2 \left(2MH + \frac{c_1}{4M^2} \right)^2}.$$

If $\Sigma_{H,c_2,\bar{c}_2}^2$ and $\Sigma_{H,c_1,\bar{c}_1}^1$ join smoothly, the corresponding function $f_2(r)$ satisfies

$$f'_2(r) = \frac{1}{-h} \sqrt{\frac{l_2^2}{l_2^2 - 1}} = \frac{1}{-h(r)} + \text{finite term} \quad \text{near } r = 2M,$$

and $c_2 < -8M^3H$ because the interface of region I and II is $r = 2M, t = \infty$. Calculating the limit of the finite term of $f'_2(r)$ at $r = 2M$ gives

$$\lim_{r \rightarrow 2M^-} \left(\frac{1}{-h} \sqrt{\frac{l_2^2}{l_2^2 - 1}} + \frac{1}{h} \right) = \frac{1}{2 \left(-2MH - \frac{c_2}{4M^2} \right)^2}.$$

Hence the necessary condition to join these two hypersurfaces is

$$\frac{1}{2 \left(2MH + \frac{c_1}{4M^2} \right)^2} = \frac{1}{2 \left(-2MH - \frac{c_2}{4M^2} \right)^2} \Rightarrow c_2 = c_1 \quad \text{or} \quad c_2 = -c_1 - 16M^3H.$$

Because $c_1 < -8M^3H$ and $c_2 < -8M^3H$, it follows that $c_2 = c_1$ is the only choice.

Similarly, if $\Sigma_{H,c_2,\bar{c}_2}^2$ and $\Sigma_{H,c_3,\bar{c}_3}^3$ join smoothly, their corresponding functions $f_2(r)$ and $f_3(r)$ near $r = 2M$ satisfy

$$f'_2(r) = \frac{1}{h} \sqrt{\frac{l_2^2}{l_2^2 - 1}} = \frac{1}{h(r)} + \text{finite term}, \quad f'_3(r) = \frac{1}{h} \sqrt{\frac{l_3^2}{l_3^2 - 1}} = \frac{1}{h(r)} + \text{finite term},$$

and $c_3 < -8M^3H$ because the interface of region II and I' is $r = 2M, t = -\infty$. Limits of the finite term of $f'_2(r)$ and $f'_3(r)$ at $r = 2M$ are

$$\frac{-1}{2 \left(-2MH - \frac{c_2}{4M^2} \right)^2} \quad \text{and} \quad \frac{-1}{2 \left(-2HM - \frac{c_3}{4M^2} \right)^2},$$

respectively, so the necessary condition to join $\Sigma_{H,c_2,\bar{c}_2}^2$ and $\Sigma_{H,c_3,\bar{c}_3}^3$ is

$$\frac{-1}{2 \left(-2MH - \frac{c_2}{4M^2} \right)^2} = \frac{-1}{2 \left(-2MH - \frac{c_3}{4M^2} \right)^2} \Rightarrow c_3 = c_2 \quad \text{or} \quad c_3 = -c_2 + 16M^3H.$$

Hence $c_3 = c_2$ is the only possibility because $c_3 < -8M^3H$ and $c_2 < -8M^3H$.

Next, we find relations of \bar{c}_1, \bar{c}_2 and \bar{c}_3 . Notice that for $\Sigma_{H,c_1,\bar{c}_1}^1$ and $\Sigma_{H,c_1,\bar{c}_2}^2$, they have expressions

$$f'_1(r) = -\frac{1}{h(r)} + \bar{f}'(r) \quad \text{and} \quad f'_2(r) = -\frac{1}{h(r)} + \bar{f}'(r) \quad \text{when } f'_2(r) > 0,$$

where $\bar{f}'(r)$ is as in (26). The function $\bar{f}'(r)$ comes from the finite term of $f'_1(r)$ and $f'_2(r)$ near $r = 2M$, and it is clearly well-defined at $r = 2M$. In addition, when $r > r''$ and $r \neq 2M$, $\bar{f}'(r)$ is the sum of two smooth functions. So $\bar{f}'(r)$ is finite valued for all $r > r''$.

Since we hope $\Sigma_{H,c_1,\bar{c}_1}^1$ and $\Sigma_{H,c_1,\bar{c}_2}^2$ to join smoothly, they must satisfy the following

condition in null coordinates:

$$\begin{aligned}
\lim_{r \rightarrow 2M^+} V(r) &= \lim_{r \rightarrow 2M^-} V(r) \\
&\Rightarrow \exp \left(\frac{1}{4M} \left(\int_{r_1}^{2M} \bar{f}'(r) dr + \bar{c}_1 \right) \right) \\
&= \exp \left(\frac{1}{4M} \left(\int_{r_2}^{2M} \bar{f}'(r) dr + \bar{c}_2 + r_2 + 2M \ln |r_2 - 2M| \right) \right) \\
&\Rightarrow \bar{c}_2 = \bar{c}_1 + \int_{r_1}^{r_2} \bar{f}'(r) dr - (r_2 + 2M \ln |r_2 - 2M|).
\end{aligned}$$

From Proposition 8, when we take $r_2 = r'_2 = r''$ and $\bar{c}'_2 = \bar{c}_2$ in (12) and (13), it follows that $\Sigma_{H,c_1,\bar{c}_2}^2 = (f_2^*(r; H, c_1, \bar{c}_2) \cup f_2^{**}(r; H, c_1, \bar{c}_2), r, \theta, \phi)$ is a complete SS-CMC hypersurfaces in region Π .

If $\Sigma_{H,c_1,\bar{c}_2}^2$ and $\Sigma_{H,c_1,\bar{c}_3}^3$ join smoothly and $r_3 = r_1$, since their expressions are

$$f'_2(r) = \frac{1}{h(r)} - \bar{f}'(r) \text{ when } f'_2(r) < 0 \quad \text{and} \quad f'_3(r) = \frac{1}{h(r)} - \bar{f}'(r),$$

they must satisfy

$$\begin{aligned}
\lim_{r \rightarrow 2M^-} U(r) &= \lim_{r \rightarrow 2M^+} U(r) \\
&\Rightarrow -\exp \left(-\frac{1}{4M} \left(-\int_{r''}^{2M} \bar{f}'(r) dr + \bar{c}_2 - r'' - 2M \ln |r'' - 2M| \right) \right) \\
&= -\exp \left(-\frac{1}{4M} \left(-\int_{r_3}^{2M} \bar{f}'(r) dr + \bar{c}_3 \right) \right) \\
&\Rightarrow \bar{c}_3 = \bar{c}_2 - \int_{r''}^{r_3} \bar{f}'(r) dr - (r'' + 2M \ln |r'' - 2M|) \\
&\Rightarrow \bar{c}_3 = \bar{c}_1 - 2 \int_{r''}^{r_1} \bar{f}'(r) dr - 2(r'' + 2M \ln |r'' - 2M|).
\end{aligned}$$

Finally, we investigate the smoothness of these complete SS-CMC hypersurfaces. When express $\Sigma_{H,c_1,\bar{c}_1}^1$ and $\Sigma_{H,c_1,\bar{c}_2}^2$ in null coordinates near the joint point, they both have

$$\begin{aligned}
U &= (r - 2M) \exp \left(\frac{1}{4M} \left(2r - \int_{r_1}^r \bar{f}'(r) dr + r_1 + 2M \ln |r_1 - 2M| \right) \right) \\
V &= \exp \left(\frac{1}{4M} \left(\int_{r_1}^r \bar{f}'(r) dr + r_1 + 2M \ln |r_1 - 2M| \right) \right).
\end{aligned}$$

Hence $\Sigma_{H,c_1,\bar{c}_1}^1 \cup \Sigma_{H,c_1,\bar{c}_2}^2$ is smooth.

The expressions for $\Sigma_{H,c_1,\bar{c}_2}^2$ and $\Sigma_{H,c_1,\bar{c}_3}^3$ in null coordinates near the joint point are both

$$\begin{aligned}
U &= -\exp \left(-\frac{1}{4M} \left(\int_{r_1}^r \bar{f}'(r) dr + r_1 + 2M \ln |r_1 - 2M| \right) \right), \\
V &= (r - 2M) \exp \left(\frac{1}{4M} \left(2r - \int_{r_1}^r \bar{f}'(r) dr + r_1 + 2M \ln |r_1 - 2M| \right) \right).
\end{aligned}$$

Hence $\Sigma_{H,c_1,\bar{c}_2}^2 \cup \Sigma_{H,c_1,\bar{c}_3}^3$ is smooth. \square

For $c_1 > -8M^3H$, the SS-CMC hypersurfaces lie in region I, Π' , and I'. Denote $C_H = \max_{r \in (0,2M)} -Hr^3 + r^{\frac{3}{2}}(2M - r)^{\frac{1}{2}}$ and note that $C_H > -8M^3H$ from Figure 8. By similar argument, we have the following results.

Theorem 3. *Given constant mean curvature H , $c_1 > -8M^3H$, and $\bar{c}_1 \in \mathbb{R}$, it determines a SS-CMC hypersurface $\Sigma_{H,c_1,\bar{c}_1}^1$ in region I. This $\Sigma_{H,c_1,\bar{c}_1}^1$ connects smoothly with a SS-CMC hypersurface $\Sigma_{H,c_1,\bar{c}_4'}^4$ for some \bar{c}_4' in region Π' . Moreover,*

- (a) *when $c_1 > C_H$, the corresponding SS-CMC hypersurface $\Sigma_{H,c_1,\bar{c}_4'}^4$ in Schwarzschild is defined on $(0, 2M)$, and $\Sigma^1 \cup \Sigma^4$ forms a complete and smooth SS-CMC hypersurface in the Kruskal extension with two ends.*
- (b) *when $c_1 = C_H$, the corresponding SS-CMC hypersurface $\Sigma_{H,c_1,\bar{c}_4'}^4$ in Schwarzschild is defined on $(R_H, 2M)$, and $\Sigma^1 \cup \Sigma^4$ forms a complete and smooth SS-CMC hypersurface in the Kruskal extension with two ends. Here R_H is defined as in Proposition 11.*
- (c) *when $-8M^3H < c_1 < C_H$, $\Sigma_{H,c_1,\bar{c}_4'}^4$ connects smoothly to a SS-CMC hypersurface $\Sigma_{H,c_1,\bar{c}_3}^3$ for some \bar{c}_3 in I'. Then $\Sigma^1 \cup \Sigma^4 \cup \Sigma^3$ forms a complete and smooth SS-CMC hypersurface in the Kruskal extension with two ends.*

Remark 8. The followings are some descriptions for the ends in each case of Theorem 3.

- In case (a), among the two ends, one is toward the space infinity $r = \infty$ in the first Schwarzschild exterior, and the other is toward the space singularity $r = 0$ in the second Schwarzschild interior.
- In case (b), one end is toward the space infinity $r = \infty$ in the first Schwarzschild exterior, and the other end is asymptotic to a cylindrical SS-CMC hypersurface $r = r_H$ in the second Schwarzschild interior.
- In case (c), the two ends are toward space infinities $r = \infty$ in the first and second Schwarzschild exteriors, respectively.

Theorem 4. *Consider $c_1 > -8M^3H$ and complete SS-CMC hypersurfaces which are described in Theorem 3. Take $r_1 > 2M$ satisfying $r_1 + 2M \ln |r_1 - 2M| = 0$ and denote $\tilde{f}'(r) = f_1'(r) - \frac{1}{h(r)}$, which can be expressed as*

$$\tilde{f}'(r) = \frac{-r^4}{(Hr^3 + c_1)^2 + r^3(r - 2M) + (Hr^3 + c_1)\sqrt{(Hr^3 + c_1)^2 + r^3(r - 2M)}},$$

Then

$$\bar{c}'_4 = \bar{c}_1 + \int_{r'_4}^{r_1} \tilde{f}'(r) dr + (r'_4 + 2M \ln |r'_4 - 2M|),$$

where $r'_4 < 2M$ is in the domain of $f_4(r)$. In case (c) of Theorem 3, if we take $r'_4 = r_4 = r''$, $\bar{c}'_4 = \bar{c}_4$, and $r_3 = r_1$ in (23) and (24), then

$$\begin{aligned} \bar{c}_3 &= \bar{c}'_4 + \int_{r''}^{r_3} \tilde{f}'(r) dr + (r'' + 2M \ln |r'' - 2M|) \\ &= \bar{c}_1 + 2 \int_{r''}^{r_1} \tilde{f}'(r) dr + 2(r'' + 2M \ln |r'' - 2M|). \end{aligned}$$

When $c_1 = -8M^3H$, we have

Theorem 5. Given constant mean curvature H , $c_1 = -8M^3H$, and $\bar{c}_1 \in \mathbb{R}$, it determines a SS-CMC hypersurface $\Sigma_{H,c_1,\bar{c}_1}^1$ in region I. This $\Sigma_{H,c_1,\bar{c}_1}^1$ connects with a SS-CMC hypersurface $\Sigma_{H,c_1,\bar{c}_3}^3$ for some \bar{c}_3 in region I' such that $\Sigma^1 \cup \Sigma^3$ forms a complete and smooth SS-CMC hypersurface in the Kruskal extension with two ends.

Remark 9. In Theorem 5, the two ends are toward space infinities $r = \infty$ in the first and the second Schwarzschild exteriors, respectively.

Proof. When $c_1 = c_3 = -8M^3H$, both $\Sigma_{H,c_1,\bar{c}_1}^1$ and $\Sigma_{H,c_3,\bar{c}_3}^3$ pass through the origin in the Kruskal extension which corresponds to $r = 2M$ with finite t . Now we determine the relation between \bar{c}_1 and \bar{c}_3 . Since

$$\begin{cases} U(r) = \sqrt{r-2M} \exp \left(\frac{1}{4M} \left(-\int_{r_1}^r f'_1(x) dx - \bar{c}_1 + r \right) \right) \\ V(r) = \sqrt{r-2M} \exp \left(\frac{1}{4M} \left(\int_{r_1}^r f'_1(x) dx + \bar{c}_1 + r \right) \right) \end{cases} \quad \text{in region I,}$$

$$\begin{cases} U(r) = -\sqrt{r-2M} \exp \left(\frac{1}{4M} \left(-\int_{r_3}^r f'_3(x) dx - \bar{c}_3 + r \right) \right) \\ V(r) = -\sqrt{r-2M} \exp \left(\frac{1}{4M} \left(\int_{r_3}^r f'_3(x) dx + \bar{c}_3 + r \right) \right) \end{cases} \quad \text{in region I' ,}$$

we have

$$\begin{aligned} \left. \frac{dV}{dU} \right|_{U=0} &= \left. \frac{\frac{dV}{dr}}{\frac{dU}{dr}} \right|_{r=2M} = \exp \left(\frac{1}{2M} \left(\int_{r_1}^{2M} f'_1(r) dr + \bar{c}_1 \right) \right) \quad \text{in region I,} \\ \left. \frac{dV}{dU} \right|_{U=0} &= \exp \left(\frac{1}{2M} \left(\int_{r_3}^{2M} f'_3(r) dr + \bar{c}_3 \right) \right) \quad \text{in region I' .} \end{aligned}$$

The integrals $\int_{r_1}^{2M} f'_1(r) dr$ and $\int_{r_3}^{2M} f'_3(r) dr$ are finite because both $f'_1(r)$ and $f'_3(r)$ are of order $O((r-2M)^{-\frac{1}{2}})$ when $H \neq 0$ and are 0 when $H = 0$.

The condition to form a C^1 complete SS-CMC hypersurface is

$$\bar{c}_3 = \bar{c}_1 + \int_{r_1}^{2M} f'_1(r)dr - \int_{r_3}^{2M} f'_3(r)dr.$$

We can take $r_1 = r_3 = 2M$, and it gives $\bar{c}_1 = \bar{c}_3$.

Now we show smoothness of the SS-CMC hypersurface. When $H = 0$, $\Sigma^1 \cup \Sigma^3$ is a straight line through the origin point in the Kruskal extension, so it is smooth. When $H \neq 0$, we have $f'_1(r) = h^{-\frac{1}{2}}(r)F_1(r)$ and $f'_3(r) = h^{-\frac{1}{2}}(r)F_3(r)$, where $F_1(r)$ and $F_3(r)$ are smooth functions on $r \geq 2M$ and $F_1(2M) = -F_3(2M)$. Furthermore, by taking $r_1 = r_3 = 2M$ and $\bar{c}_1 = \bar{c}_3$, we have

$$\frac{dV}{dU} = \begin{cases} \exp\left(\frac{1}{2M} \left(\int_{2M}^r f'_1(x)dx + \bar{c}_1\right)\right) \left(1 + \frac{2h^{\frac{1}{2}}(r)F_1(r)}{1-h^{\frac{1}{2}}(r)F_1(r)}\right) & \text{in region I} \\ \exp\left(\frac{1}{2M} \left(\int_{2M}^r f'_3(x)dx + \bar{c}_1\right)\right) \left(1 + \frac{2h^{\frac{1}{2}}(r)F_3(r)}{1-h^{\frac{1}{2}}(r)F_3(r)}\right) & \text{in region I'}. \end{cases}$$

Because

$$\frac{d^2V}{dU^2} = \frac{dr}{dU} \left(\frac{d}{dr} \frac{dV}{dU} \right),$$

it gives

$$\lim_{r \rightarrow 2M} \frac{d^2V}{dU^2} = \begin{cases} 2\sqrt{2M} \frac{F_1(2M)}{M} \exp\left(\frac{1}{4M} (3\bar{c}_1 - 2M)\right) & \text{in region I} \\ -2\sqrt{2M} \frac{F_3(2M)}{M} \exp\left(\frac{1}{4M} (3\bar{c}_1 - 2M)\right) & \text{in region I'}. \end{cases}$$

Since $F_1(2M) = -F_3(2M)$, we get the SS-CMC hypersurface is C^2 .

When we rewrite the SS-CMC hypersurface in the coordinates $(T = F(X), X, \theta, \phi)$, the SS-CMC equation becomes

$$\begin{aligned} F''(X) + e^{-\frac{r}{2M}} \left(\frac{6M}{r^2} - \frac{1}{r} \right) (-F(X) + F'(X)X)(1 - (F'(X))^2) \\ + \frac{12HM e^{-\frac{r}{4M}}}{\sqrt{r}} (1 - (F'(X))^2)^{\frac{3}{2}} = 0, \end{aligned}$$

where the spacelike condition is $1 - (F'(X))^2 > 0$ and r is considered as a function of X and $T = F(X)$ by (3). Once we know that the SS-CMC hypersurface is C^2 , the standard PDE theory (see [2, Theorem 6.17.] for example) implies that the SS-CMC hypersurface is C^∞ . \square

Figures 9 and 10 show some complete SS-CMC hypersurfaces in the Kruskal extension for $H > 0$ and $H < 0$, respectively.

Remark 10. Besides complete SS-CMC hypersurfaces in Theorems 1 and 3, there are different SS-CMC hypersurfaces in the Kruskal extension as below:

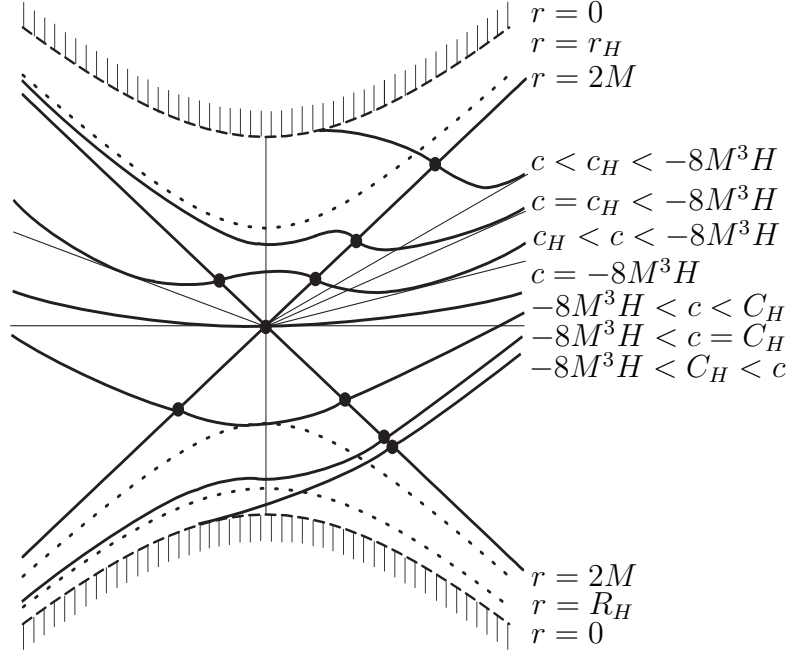


Figure 9: SS-CMC hypersurfaces with $H > 0$.

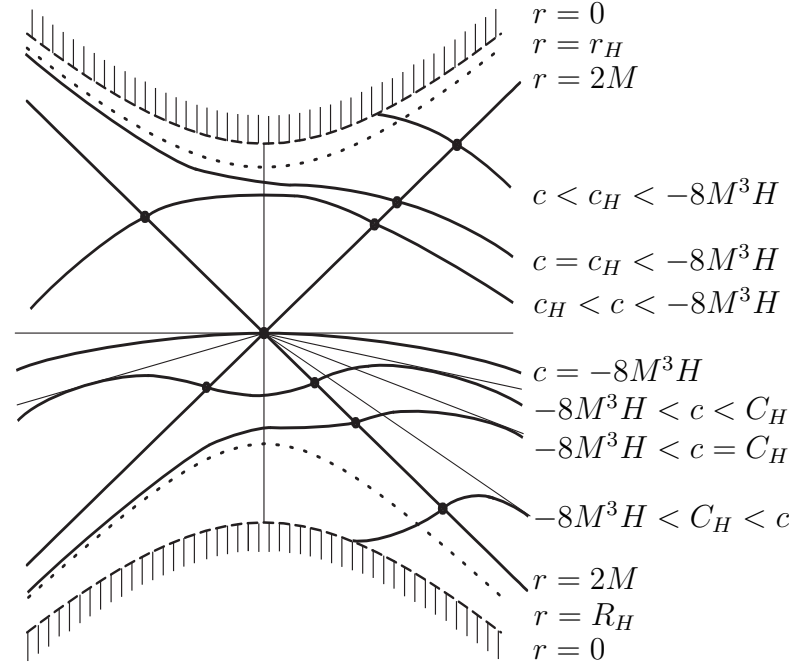


Figure 10: SS-CMC hypersurfaces with $H < 0$.

- In case (b) of Proposition 8 and $f_2(r)$ is defined on $(0, r_H)$, the SS-CMC hypersurface is mapped to the region Π with two ends. One of them is toward the space singularity $r = 0$, and the other is toward the cylindrical hypersurface $r = r_H$.
- In case (c) of Proposition 8 and $f_2(r)$ is defined on $(0, r']$, the SS-CMC hypersurface is mapped to the region Π with two ends. These two ends are toward the space singularity $r = 0$. This SS-CMC hypersurface is C^∞ .
- In case (b) of Proposition 12 and $f_4(r)$ is defined on $(0, R_H)$, the SS-CMC hypersurface is mapped to the region Π' with two ends. One of them is toward the space singularity $r = 0$, and the other is toward the cylindrical hypersurface $r = R_H$.
- In case (c) of Proposition 12 and $f_4(r)$ is defined on $(0, r']$, the SS-CMC hypersurface is mapped to the region Π' with two ends. These two ends are toward the space singularity $r = 0$. This SS-CMC hypersurface is C^∞ .
- We can also start with a SS-CMC hypersurface $\Sigma_{H, c_3, \bar{c}_3}^3$ in region I' , and apply similar arguments as Theorems 1 and 3. New complete and smooth SS-CMC hypersurfaces can be found. They correspond to the case (a), (b) of Theorem 1 and 3, which all have one end toward space infinity $r = \infty$ in the second Schwarzschild exterior. Their another end can be toward space singularity $r = 0$ (case (a)) or cylindrical hypersurface (case(b)) in either the first or the second Schwarzschild interior.

Theorem 6. *All complete SS-CMC hypersurfaces in the Kruskal extension are as in Theorems 1, 3, 5, Remark 10, and cylindrical hypersurfaces.*

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Kuo-Wei Lee

INSTITUTE OF MATHEMATICS, ACADEMIA SINICA, TAIPEI, TAIWAN

E-mail address: d93221007@gmail.com

Yng-Ing Lee

DEPARTMENT OF MATHEMATICS, NATIONAL TAIWAN UNIVERSITY, TAIPEI, TAIWAN

NATIONAL CENTER FOR THEORETICAL SCIENCES, TAIPEI OFFICE, NATIONAL TAIWAN UNIVERSITY, TAIPEI, TAIWAN

E-mail address: yilee@math.ntu.edu.tw